Chapter II Some spaces of distributions and operators

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **17 (1971)**

Heft 1: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **13.07.2024**

pactification for which $B_{x,w}$ has a unique minimal element. It may be possible to obtain such results by arguments of the type used by Gabrielov [1] to prove that for every P the closed union of all $\Lambda'(Q)$, $Q \in L(P)$, is a semi-algebraic set of codimension at least one.

For other definitions of the wave front set we refer to Sato [1, 2], and Sato and Kashiwara [1] for the case of hyperfunctions relative to real analytic functions, and to Hörmander [11] for the case of Schwartz distributions relative to any Denjoy-Carleman class of functions which is closed under differentiation and contains the real analytic functions.

Chapter II

Some spaces of distributions and operators

2.1. *Pseudo-differential operators*

In Chapter I all results ultimately depended on the Fourier transformation. When the coefficients are variable we need to have some substitute. The simplest case occurs in the construction of fundamental solutions for *elliptic* operators with variable coefficients. Classically this was done by perturbation arguments (the E. E. Levi parametrix method, Korn's approximation). These ideas are now embedded in a more manageable and precise form in the theory of pseudo-differential operators.

Let us first note that for an elliptic operator $P(D)$ with constant coefficients of order m we have for some constant C,

$$|\xi|^m \leqq C\,|P(\xi)|\,,\quad |\xi| > C\,,$$

if ξ is real or belongs to a narrow cone in $\mathbf{C}^n$ containing $\mathbf{R}^n$. Apart from an integration over a compact set, which contributes an entire analytic term, the fundamental solution constructed in section 1.1 is therefore simply

$$Ef(x) = (2\pi)^{-n} \int e^{i<x,\xi>}\, \chi(\xi)/P(\xi)\hat{f}(\xi)\, d\xi\,.$$

Here χ is a fixed C^∞ function which is 0 when $|\xi| < C$ and 1 for large $|\xi|$. Differentiation under the sign of integration gives, with E also denoting the distribution such that $Ef = E * f$,

$$P(D)E = \delta + R. \tag{2.1.1}$$

Here $\hat{R} = \chi - 1$ so that $R \in C^\infty$. One calls E a *parametrix.* Outside the origin we have $E \in C^\infty$, for if α is large then

$$(-x)^\alpha E = (2\pi)^{-n} \int e^{i<x,\xi>} D_\xi^\alpha \big(\chi(\xi)/P(\xi)\big)\, d\xi\,,$$

and the integrand decreases rapidly at infinity. For the study of regularity properties it is as useful to have a parametrix as to have a fundamental solution: If $v \in \mathscr{E}'$ we obtain $v = E * \big(P(D)v\big) - R * v$. Here $R * v \in C^\infty$ and $E * P(D)v \in C^\infty$ outside sing supp $P(D)v$ since $E \in C^\infty$ outside the origin. This gives

$$\text{sing supp}\, v = \text{sing supp}\, P(D)v$$

when v has compact support and therefore for arbitrary v.

Consider now a differential operator P with variable coefficients,

$$P(x, D) = \sum_{|\alpha| \leqq m} a_\alpha(x) D^\alpha$$

in an open set $X \subset \mathbf{R}^n$. We assume that $a_\alpha \in C^\infty(X)$ and that P is elliptic in X, that is,

$$P_m(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x)\xi^\alpha \neq 0 \quad \text{if } x \in X \text{ and } 0 \neq \xi \in \mathbf{R}^n.$$

We want to construct a (right) parametrix E, that is, a linear map $C_0^\infty(X) \to$ $\to C^\infty(X)$ such that $P(x, D)E = I + R$ where R is an integral operator with C^∞ kernel. The classical method of E. E. Levi is to take a fixed $x_0 \in X$ and try to find E as a perturbation of the known parametrix of the operator $P(x_0, D)$, that is,

$$Ef(x) = (2\pi)^{-n} \int e^{i<x,\xi>} \chi(\xi)/P(x_0,\xi)\hat{f}(\xi)\, d\xi\,.$$

Naturally this must be a better approximation at x_0 than elsewhere, so the approximation is improved if one replaces $P(x_0, \xi)$ by $P(x, \xi)$. Note that $P(x, \xi)^{-1} = P_m(x, \xi)^{-1} + \ldots$ where dots indicate homogeneous terms of order $-m-1$, $-m-2$, ... Thus we are led to consider operators of the form

$$Ef(x) = (2\pi)^{-n} \int e^{i<x,\xi>} e(x, \xi)\hat{f}(\xi)\, d\xi \tag{2.1.2}$$

where e behaves asymptotically when $\xi \to \infty$ as a sum of homogeneous functions of ξ. (See Kohn-Nirenberg [1], Hörmander [2, 4] and the references given there.) Actually it is preferable to make the somewhat less restrictive assumption that $e \in C^\infty(X \times \mathbf{R}^n)$ and that for some μ and all multi-indices α and β

$$(2.1.3) \qquad |D_\xi^\alpha D_x^\beta e(x, \xi)| \leqq C_{\alpha,\beta,K}(1+|\xi|)^{\mu-|\alpha|}, \quad x \in K \subset\subset X.$$

The set of all such functions will be denoted by $S^\mu(X \times \mathbf{R}^n)$. An operator of the form (2.1.2) with $e \in S^\mu$ is called a *pseudo-differential operator* of order μ with symbol e. It is easy to see that E is a continuous map from $C_0^\infty(X)$ to $C^\infty(X)$ and that E can be extended to a continuous map from $\mathscr{E}'(X)$ to $\mathscr{D}'(X)$. The diagonal in $X \times X$ contains all singularities of the kernel of E (which is a distribution in $X \times X$). Summing up these facts one finds the pseudo-local property

$$(2.1.4) \qquad \text{sing supp } Eu \subset \text{sing supp } u, u \in \mathscr{E}'(X).$$

To complete the construction of a parametrix for the elliptic operator P it suffices to choose e so that

$$P(x, D+\xi)\,e(x,\xi) - 1 \in S^{-\infty} = \bigcap_\mu S^\mu.$$

To do so we choose e asymptotic to a sum $e_0 + e_1 + \dots$ where e_j is homogeneous of degree $-m-j$ with respect to ξ and

$$P(x, D+\xi)(e_0 + \dots + e_j) - 1 \in S^{-j-1}, \; j = 0, 1, \dots .$$

This means for $j = 0$ that $e_0 = 1/P_m$. Since $P(x, D+\xi)\,e_j - P_m(x,\xi)\,e_j \in$ $\in S^{-j-1}$ the conditions are recursively satisfied by a suitable choice of e_j. This formal successive approximation is of course just a simpler way of carrying out the classical iterative procedures for solving the integral equations which occur in the E. E. Levi method. It is more appropriate though, since it avoids strict convergence requirements which force one to work locally only.

Pseudo-differential operators not only give a convenient framework for the construction of parametrices for elliptic equations but they form a natural extension of the class of differential operators. A differential operator $P(x, D)$ is obviously of the form (2.1.2) with $e(x, \xi) = P(x, \xi)$. It turns out that also pseudo-differential operators form an *algebra which is invariant under passage to adjoints and changes of variables*; the latter fact immediately allows an extension of the definition to manifolds. The usual formulas of calculus remain valid with obvious modifications. For example, if $P =$ $= P(x, D)$ and $Q = Q(x, D)$ are differential operators then the symbol of the differential operator $R = QP$ is given by

$$(2.1.5) \qquad R(x, \xi) = \sum \left((iD_\xi)^\alpha Q(x, \xi) \right) D_x^\alpha P(x, \xi)/\alpha\,!$$

If P and Q are pseudo-differential operators with symbols $P(x, \xi)$, $Q(x, \xi)$ the product $R = QP$ is again a pseudo-differential operator and for the symbol $R(x, \xi)$ the formula (2.1.5) is valid *mod* S^μ for every μ, which makes sense since all but a finite number of terms are in S^μ. One precaution must be made though, for to compose pseudo-differential operators we must assume that they map C_0^∞ to C_0^∞, and preferably also C^∞ to C^∞. Since the kernel of a pseudo-differential operator is in C^∞ outside the diagonal in $X \times X$ it can be modified without changing the singularities to a kernel K with support so close to the diagonal that the projections $\operatorname{supp} K \to X$ are both proper. This implies the desired properties. We shall say that an operator with such a kernel is *properly supported.* By $L^\mu(X)$ we denote the space of properly supported pseudo-differential operators of order μ. The definition is clearly valid also if X is a C^∞ manifold.

Generalizing a definition in section 1.3 for differential operators we shall say that a pseudo-differential operator P of order m with symbol p is characteristic at $(x, \xi) \in X \times (\mathbf{R}^n \backslash 0)$ if

$$\varliminf_{t \to +\infty} \; |p(x, t\xi)|\, t^{-m} = 0\,.$$

The characteristic points form a closed cone in $X \times (\mathbf{R}^n \backslash 0)$ which regarded as a subset of $T^*(X) \backslash 0$ is invariant under a change of variables and therefore well defined even if X is a manifold. If no characteristic exists, we say that P is *elliptic*. The arguments above show that if P is elliptic of order m one can find Q elliptic of order $-m$ so that $QP - I = R_1$ and $PQ - I = R_2$ have C^∞ kernels. This shows that also for elliptic pseudo-differential operators we have

$$\operatorname{sing\,supp} u = \operatorname{sing\,supp} Pu, \; u \in \mathcal{D}'(X)\,.$$

The construction of fundamental solutions in section 1.1 also simplifies very much when P is just hypoelliptic, that is, P satisfies (1.4.2). This condition implies that $P(\xi) \neq 0$ for large ξ and that for some $\rho > 0$

$$|D_\xi^\alpha P(\xi)| / |P(\xi)| \leqq C |\xi|^{-\rho|\alpha|},$$

One can still find a parametrix of the form (2.1.2), but $e(x, \xi) = 1/P(\xi)$ satisfies a weaker condition than (2.1.3). One is therefore led to introduce the set $S_{\rho,\delta}^m$ of functions such that for all multi-indices

$$(2.1.3)' \quad |D_\xi^\alpha D_x^\beta e(x, \xi)| \leqq C_{\alpha,\beta,K} (1 + |\xi|)^{m - \rho|\alpha| + \delta|\beta|}, \; x \in K \subset\subset X\,.$$

When $0 \leqq \delta < \rho \leqq 1$ one obtains again a self adjoint algebra of operators, and it is invariant under a change of variables if in addition $1 - \rho \leqq \delta$. If for some $\delta < \rho$

$$|D_\xi^\alpha D_x^\beta p(x, \xi)| / |p(x, \xi)| \leqq C_{\alpha,\beta,K}(1 + |\xi|)^{-\rho|\alpha|+\delta|\beta|}, \; x \in K ((X,$$

and $1/|p(x, \xi)| \leqq C |\xi|^M$ for some M, one can as in the elliptic case construct a parametrix of the same type and conclude that the operator with symbol p is hypoelliptic. (See Hörmander [4] and for the case of systems also Hörmander [8].) However, for the sake of brevity we shall ignore extensions of this type in what follows.

If $L^2_{loc}(X)$ is the set of functions in X which are square integrable on compact subsets of every coordinate patch (with the obvious topology), then every $P \in L^0(X)$ is a continuous map $L^2_{loc}(X) \to L^2_{loc}(X)$. If we define $H_{(s)}(X)$ to be the set of all distributions such that $Pu \in L^2_{loc}(X)$ when $P \in L^s(X)$ it follows that $H_{(0)}(X) = L^2_{loc}(X)$, and that $L^m(X)$ maps $H_{(s)}(X)$ continuously into $H_{(s-m)}(X)$. Conversely $Pu \in H_{(s-m)}(X)$ implies $u \in H_{(s)}(X)$ if $P \in L^m(X)$ is elliptic, so $u \in H_{(s)}(X)$ if (and only if) $Pu \in L^2_{loc}(X)$ for some elliptic P of order s. Similar definitions can be made with L^2 replaced by L^p if $1 < p < \infty$.

2.2. *The wave front set*

If $u \in \mathscr{D}'(X)$ we have by definition

$$\text{sing supp } u = \cap \{x; \varphi(x) = 0\}$$

the intersection being taken over all $\varphi \in C^\infty(X)$ with $\varphi u \in C^\infty(X)$. Replacing the function φ by a pseudo-differential operator A we introduce

$$WF(u) = \bigcap_{Au \in C^\infty} char(A) \tag{2.2.1}$$

where *char* (A) is the set of characteristics of A. It is clear that this is a closed cone in $T^*(X)\backslash 0$ with projection in X contained in sing supp u. In fact, it is equal to sing supp u for if x is not in the projection of $WF(u)$ we can find finitely many operators $A_j \in L^0$ with $A_j u \in C^\infty$ so that $T^*_x \cap (\cap\, char(A_j)) = \varnothing$. If $A = \Sigma A_j^* A_j$ we have $Au \in C^\infty$ and A is elliptic at x so $u \in C^\infty$ there. Thus we have

THEOREM 2.2.1. *The projection of $WF(u)$ in X is equal to* sing supp u.

We shall call $WF(u)$ the *wave front set* of u. (The relation to the definitions in section 1.6 will be discussed after Theorem 2.2.3.) Clearly it describes the location of the singularities and the frequencies which occur in their harmonic decomposition. The definition we have given leads immediately to a regularity theorem for any pseudo-differential operator:

THEOREM 2.2.2. *If A is a pseudo-differential operator then*

$$WF(Au) \subset WF(u) \subset WF(Au) \cup char(A). \tag{2.2.2}$$

Proof. The second part, extending the regularity theorem for elliptic operators is obvious, but the first which improves the pseudolocal property (2.1.4) may require some comment. We may assume that $X \subset \mathbf{R}^n$ since the definition of $WF(u)$ is local in X. For any $(x_0, \xi_0) \notin WF(u)$ we can choose a pseudo-differential operator B which is non-characteristic at (x_0, ξ_0) so that $Bu \in C^\infty$. If C is a pseudo-differential operator whose symbol is of order $-\infty$ outside a small conic neighborhood of (x_0, ξ_0) we can find another operator C_1 such that $CA = C_1 B$, by multiplying CA to the right with the formal inverse of B which exists near (x_0, ξ_0). Thus $CAu = C_1 Bu \in$ $\in C^\infty$ and we conclude that $(x_0, \xi_0) \notin WF(Au)$.

In the definition of the wave front set it is easily seen that one can restrict oneself to operators A of order 0 and even operators of the form $b(D)\,a(x)$ where $b(\xi)$ is a homogeneous function of degree 0 for large $|\xi|$. This leads to an equivalent definition which is more useful in many proofs:

THEOREM 2.2.3. $(x_0, \xi_0) \notin WF(u)$ *if and only if for some coordinate patch containing x_0 one can find $v \in \mathscr{E}'$ equal to u in a neighborhood of x_0 and with $\hat{v}(\xi) = 0(|\xi|^{-N})$ for every N in a conic neighborhood of ξ_0 independent of N.*

The theorem shows that $WF(u)$ regarded as a subset of the sphere bundle agrees with the set given by Definition 1.6.3 when $X \subset \mathbf{R}^n$ and W_0 is the compactification by a sphere. The definition used here has the advantage that the invariance under a change of variables follows from the invariance of pseudo-differential operators.

We shall now list a number of properties of wave front sets. Most of them are due to Sato who considered hyperfunctions modulo real analytic functions. (See Sato [2], Sato-Kawai [1], and Sato-Kashiwara [1].) For complete proofs using Theorem 2.2.3 see Hörmander [9, section 2.5].

First we consider the product of two distributions u_1 and u_2. Let $\chi \in C_0^\infty(\mathbf{R}^n)$, $\int \chi dx = 1$, and set $\chi_\varepsilon(x) = \varepsilon^{-n}\chi(x/\varepsilon)$. Assuming that $u_j \in \mathscr{E}'(\mathbf{R}^n)$ we wish to define $u_1 u_2$ as the limit of $(u_1 * \chi_\varepsilon)(u_2 * \chi_\varepsilon)$ as $\varepsilon \to 0$. In general this is not possible but the limit does exist if

$$(2.2.3) \qquad WF(u_1) + WF(u_2) = \{(x, \xi_1 + \xi_2);\ (x, \xi_j) \in WF(u_j)\} \subset T^*(X)\backslash 0$$

It is then independent of the choice of coordinates and χ. The situation is summed up in

THEOREM 2.2.4. *If $u_1, u_2 \in \mathscr{D}'(X)$ and* (2.2.3) *is fulfilled, there is a natural way of defining $u_1 u_2$ and we have*

$$(2.2.4) \qquad WF(u_1 u_2) \subset WF(u_1) \cup WF(u_2) \cup \big(WF(u_1) + WF(u_2)\big).$$

Here the right hand side is closed and X may be a manifold.

With suitable definitions the multiplication is continuous when introduced in this way. In the following theorems the word " natural " will refer to a definition by continuity as in Theorem 2.2.4.

THEOREM 2.2.5. *Let X and Y be manifolds and $\varphi : Y \to X$ a C^∞ map. Let $u \in \mathscr{D}'(X)$ and assume that*

$$\varphi^* WF(u) = \{(y, {}^t\varphi_y'(y)\,\xi), (\varphi(y), \xi) \in WF(u)\} \subset T^*(Y)\backslash 0 .$$

Then there is a natural way of defining the composition $\varphi^* u$ of u with φ so that it is the standard composition when u is a function. We have

$$(2.2.5) \qquad WF(\varphi^* u) \subset \varphi^* WF(u).$$

Note that the pullback $\varphi^* u$ is defined for all $u \in \mathscr{D}'(X)$ precisely when φ' is surjective, and then it is well known that such a definition is possible. In particular we see that if $Y \subset X$ is a submanifold, we can define the restriction of u to Y if the normal bundle $N(Y)$ does not meet $WF(u)$. For example, if $u \in \mathscr{D}'(X)$ and $Au \in C^\infty$ for some pseudo-differential operator A, we can define the restriction of u to Y if Y is non-characteristic, that is, the normals to Y are non-characteristic with respect to A. This is also a well known fact (partial hypoellipticity).

Let X and Y be two C^∞ manifolds with given positive C^∞ densities. By the kernel theorem of Schwartz we can then identify $\mathscr{D}'(X \times Y)$ with the space of continuous linear operators $C_0^\infty(Y) \to \mathscr{D}'(X)$ by means of the formula

$$< K\varphi, \psi > \ = K(\psi \otimes \varphi); \ \varphi \in C_0^\infty(Y), \psi \in C_0^\infty(X);$$

on the right K denotes an element of $\mathscr{D}'(X \times Y)$ and on the left the corresponding linear transformation. In terms of the wave front set of K we can state useful sufficient conditions for regularity of K in the sense of Schwartz [1]:

THEOREM 2.2.6. *For any $u \in C_0^\infty(Y)$ the set*

$$(2.2.6) \quad WF_X(K) = \{(x, \xi); (x, \xi, y, 0) \in WF(K) \text{ for some } y \in Y\}$$

contains $WF(Ku)$. Thus K maps $C_0^\infty(Y)$ into $C^\infty(X)$ if $WF_X(K) = \varnothing$, that is, if $WF(K)$ contains no point which is normal to a manifold $x =$ constant.

THEOREM 2.2.7. *Ku can be defined in a natural way when $u \in \mathscr{E}'(Y)$ and $WF(u)$ does not meet the set*

$$(2.2.7) \quad WF'_Y(K) = \{(y, \eta); (x, 0, y, -\eta) \in WF(K) \text{ for some } x \in X\}.$$

Thus K can be extended to a continuous map $\mathscr{E}'(Y) \to \mathscr{D}'(X)$ if $WF'_Y(K) = \varnothing$, that is, $WF(K)$ contains no point which is normal to a manifold $y =$ constant.

The proof of Theorem 2.2.6 follows easily from the description of the wave front set given in Theorem 2.2.3. Theorem 2.2.7 follows by duality.

If we have three manifolds X, Y, Z and distributions $K_1 \in \mathscr{D}'(X \times Y)$, $K_2 \in \mathscr{D}'(Y \times Z)$ where for simplicity we assume that K_1 and K_2 are properly supported, then $K_2 u \in \mathscr{E}'(Y)$ and $WF(K_2 u) \subset WF_Y(K_2)$ when $u \in C_0^\infty(Z)$. The composition $K_1(K_2 u)$ is therefore defined if

$$(2.2.8) \qquad WF'_Y(K_1) \cap WF_Y(K_2) = \varnothing,$$

and it is of the form $(K_1 \circ K_2) u$ where $K_1 \circ K_2 \in \mathscr{D}'(X \times Z)$. When writing down an inclusion for the wave front set of $K_1 \circ K_2$ it is convenient to introduce for example

$$WF'(K_1) = \{(x, \xi, y, \eta); (x, \xi, y, -\eta) \in WF(K_1)\},$$

that is, multiply by -1 in the fiber of the second tangent space involved.

THEOREM 2.2.8. *When* (2.2.8) *is fulfilled we have*

$$\text{(2.2.9)}\quad WF'(K_1 \circ K_2) \subset \big(WF'(K_1) \circ WF'(K_2)\big) \cup \big(WF_X(K_1) \times Z\big) \cup \big(X \times WF'_Z(K_2)\big).$$

Here $WF'(K_1)$ *and* $WF'(K_2)$ *are composed as relations from* $T^*(Y)$ *to* $T^*(X)$ *and from* $T^*(Z)$ *to* $T^*(Y)$. *The right hand side of* (2.2.9) *is closed.*

The special case when Z reduces to a point is worth special notice:

THEOREM 2.2.9. *Let* $K \in \mathscr{D}'(X \times Y)$ *and* $u \in \mathscr{E}'(Y)$, $WF(u) \cap WF'_Y(K) = \varnothing$. *Then we have*

$$\text{(2.2.10)}\qquad WF(Ku) \subset \big(WF'(K) \circ WF(u)\big) \cup WF_X(K)$$

where again $WF'(K)$ *is interpreted as a relation mapping sets in* $T^*(Y)$ *to sets in* $T^*(X)$.

In section 2.3 we shall describe the wave front set for some important classes of distributions. In preparation for this we shall now discuss how the wave front set can be used to localize various spaces of distributions not only in X but in $T^*(X)\backslash 0$ (or rather the cosphere bundle $S^*(X)$ which is the quotient by the multiplicative group of positive reals).

Let $\mathscr{F}$ be a linear subspace of $\mathscr{D}'(X)$. If $x_0 \in X$ we shall say that a distribution u in X belongs to $\mathscr{F}$ at x_0 if one can find $v \in \mathscr{F}$ so that $v - u = 0$ in a neighborhood of x_0. We call $\mathscr{F}$ *local* if every distribution which belongs to $\mathscr{F}$ at every $x_0 \in X$ is in fact in $\mathscr{F}$. (This means that $\mathscr{F}$ is the space of sections of the sheaf of germs of sections of $\mathscr{F}$.)

If $(x_0, \xi_0) \in T^*(X)\backslash 0$ we shall say that $u \in \mathscr{F}$ at (x_0, ξ_0) if one can find $v \in \mathscr{F}$ so that $(x_0, \xi_0) \notin WF(u-v)$. Repeating the proof of Theorem 2.2.1 one shows that when $C^\infty(X) \subset \mathscr{F}$ and $\mathscr{F}$ is an L^0 *module*, then $u \in \mathscr{F}$ at x_0 if (and only if) $u \in \mathscr{F}$ at (x_0, ξ_0) for every $\xi_0 \in T^*_{x_0}\backslash 0$. If in addition $\mathscr{F}$ is local we therefore conclude that $u \in \mathscr{F}$ if and only if $u \in \mathscr{F}$ at (x_0, ξ_0) for all $(x_0, \xi_0) \in T^*(X)\backslash 0$. As an example of this we may take $\mathscr{F} = H_{(s)}(X)$.

We can also piece together spaces of distributions from local data. Let $\{ U_i \}_{i \in I}$ be a covering of $T^*(X)\backslash 0$ by open cones and let $\mathscr{F}_i$, $i \in I$, be an L^0 submodule of $\mathscr{D}'(X)$ containing $C^\infty(X)$. Assume that if $(x_0, \xi_0) \in U_i \cap U_j$ then every element of $\mathscr{F}_i$ is in $\mathscr{F}_j$ at (x_0, ξ_0). If we set

$$\mathscr{F} = \{ u \in \mathscr{D}'(X);\, u \in \mathscr{F}_j \text{ at every point in } U_j \text{ for all } j \}$$

we obtain a local L^0 module of distributions. If $(x_0, \xi_0) \in U_j$ we have $u \in \mathscr{F}$ at (x_0, ξ_0) if and only if $u \in \mathscr{F}_j$ at (x_0, ξ_0).

2.3. *Distributions defined by Fourier integrals*

If in (2.1.2) we introduce the definition of the Fourier transformation we see formally that the distribution kernel of the pseudo-differential operator E is given by

$$(x, y) \to (2\pi)^{-n} \int e^{i\langle x-y,\theta\rangle} e(x, \theta)\, d\theta . \tag{2.3.1}$$

Similarly the fundamental solution of the wave equation $\partial^2 u/\partial t^2 - \Delta u = 0$ in n space variables $(n > 1)$ with pole at $(y, 0)$ is at time $t > 0$ given by

$$(x, y) \to \tag{2.3.2}$$
$$(2\pi)^{-n} \Big(\int e^{i(\langle x-y,\theta\rangle + t|\theta|)} (2i|\theta|)^{-1}\, d\theta - \int e^{i(\langle x-y,\theta\rangle - t|\theta|)} (2i|\theta|)^{-1}\, d\theta\Big) .$$

These examples suggest the importance of the classes of distributions which we shall study now.

Let $X \subset \mathbf{R}^n$ and let Γ be an open cone in $X \times (\mathbf{R}^N \backslash 0)$ for some N. Assume given a function $\varphi \in C^\infty(\Gamma)$ satisfying the following conditions:

(i) φ is positively homogeneous with respect to the variables in $\mathbf{R}^N$.

(ii) $\operatorname{Im} \varphi \geqq 0$.

(iii) $d\varphi \neq 0$ everywhere in Γ.

Such a function will be called a *phase* function. Let $S_0^m(\Gamma)$ be the set of all $a \in S^m(X \times (\mathbf{R}^N \backslash 0))$ (see section 2.1) vanishing in a conic neighborhood of $\complement\Gamma$.

For $a \in S_0^m(\Gamma)$ we claim that the integral

$$A(x) = \int e^{i\varphi(x,\theta)} a(x, \theta)\, d\theta \tag{2.3.3}$$

can be defined, not necessarily as a function of x but as a distribution in X. To do so we consider the linear form

$$I(u) = \iint e^{i\varphi(x,\theta)} a(x, \theta)\, u(x)\, dx d\theta, u \in C_0^\infty(X) . \tag{2.3.4}$$

In view of (iii) the fact that

$$e^{i\varphi} = D(e^{i\varphi})/(Di\varphi)$$

allows one, by successive (formal) partial integrations with no boundary terms, to reduce the growth of the integrand at infinity until it becomes integrable. This gives a precise definition of $I(u)$ and the linear form $u \to I(u)$ is then a distribution $A \in \mathscr{D}'(X)$. If $\chi \in \mathscr{S}(\mathbf{R}^N)$, $\chi(0) = 1$, it is easily shown that

$$(2.3.5) \qquad A = \lim_{\varepsilon \to 0} \int e^{i\varphi(.,\theta)} \chi(\varepsilon\theta)\, a(.,\theta)\, d\theta$$

with the limit in the weak topology of $\mathscr{D}'(X)$. Thus the definition of (2.3.4) by partial integrations is quite independent of how these are carried out, and (2.3.5) is independent of the choice of χ. We shall call (2.3.3) an oscillatory integral but use the standard notation. (For these facts as well as most of this section we refer to Hörmander [9].) The integral (2.3.3) is thus defined for a fixed $x = x_0$ if $\varphi(x_0, \theta)$ has no critical point $(x_0, \theta) \in \Gamma$ as a function of θ. In that case, $A \in C^\infty$ near x_0. Note that if (x_0, θ) is a critical point of φ as a function of θ, then $\varphi(x_0, \theta) = 0$ by Euler's identity for homogeneous functions. On the other hand, when $\varphi(x_0, \theta) = 0$ it follows from (ii) that $d(\operatorname{Im} \varphi(x, \theta)) = 0$ so $d_{x,\theta} \operatorname{Re} \varphi(x, \theta) \neq 0$ by (iii).

To determine the wave front set of A we use Theorem 2.2.3. Thus we take a function $u \in C_0^\infty$ equal to 1 near x_0 and with small support, and study

$$< A, ue^{-i<x,\xi>} > = \iint e^{i(\varphi(x,\theta) - <x,\xi>)} u(x)\, a(x, \theta)\, dx d\theta$$

as $\xi \to \infty$ in a conic neighborhood of ξ_0 (oscillatory integral !). Naturally the main contributions come from critical points in the exponent, that is, points where $\varphi'_\theta = 0$, $\varphi'_x = \xi$. Indeed, we have

THEOREM 2.3.1. *If A is defined by* (2.3.5) *then*

$$(2.3.6) \qquad WF(A) \subset \{(x, \varphi'_x(x, \theta)); (x, \theta) \in \Gamma \text{ and } \varphi'_\theta(x, \theta) = 0\} \subset T^*(X) \backslash 0$$

In particular,

$$(2.3.7) \qquad \text{sing supp } A \subset \{x; \varphi'_\theta(x, \theta) = 0 \text{ for some } \theta \text{ with } (x, \theta) \in \Gamma\}.$$

As an example we see from (2.3.1) that the wave front set of the kernel of a pseudo-differential operator E lies in $\{(x, y; \xi, \eta); x = y, \xi = -\eta\}$ which is the normal bundle of the diagonal. Thus $WF'(E)$ is in the diagonal of $T^*(X) \times T^*(X)$, which allows us to identify the wave front set of a pseudo-differential operator in X with a closed cone in $T^*(X) \backslash 0$. In view of Theorem 2.2.9 this result contains the left hand part of (2.2.2) (the improved pseudo-local property).

As a second example we see that for the two terms in (2.3.2) the wave front set lies in the set where $x - y = \mp t\theta/|\theta|$ and $\xi = -\eta = \theta$.

This corresponds to the two components of the normal bundle of $\{ (x, y); | x - y |^2 = t^2 \}$. In particular the singularities are carried by the light cone.

The set of distributions which can be written in the form (2.3.5) with a given φ and $a \in S_0^m (\Gamma)$ is always an $L^0 (X)$ module. (For a proof when φ is real see Theorem 2.12 in Hörmander [6].) We can therefore use the remarks at the end of section 2.2 to define global spaces of distributions which locally in $T^*(X)\backslash 0$ have such representations.

We shall restrict ourselves in what follows to the case where φ is *real and non-degenerate*, that is, the differentials of the functions $\partial\varphi/\partial\theta_j$ are linearly independent in $C = \{ (x, \theta) \in \Gamma; \varphi'_\theta (x, \theta) = 0 \}$. The map

$$C \ni (x, \theta) \to (x, \varphi'_x) \tag{2.3.8}$$

to the wave front set has an injective differential then. The range Λ is locally a conic manifold in $T^*(X)\backslash 0$ of dimension *dim* X. Let (x, ξ) denote the standard coordinates in $T^*(X)$ obtained from local coordinates $x_1, ..., x_n$ in X by taking $dx_1, ..., dx_n$ as basis for the cotangent vectors. The form $\Sigma\xi_j \, dx_j$ is then invariantly defined, and the restriction to Λ is $\varphi'_x \, dx = d\varphi - \varphi'_\theta \, d\theta = 0$. In view of the homogeneity this is equivalent to the vanishing on Λ of the differential which is the symplectic form $\sigma = \Sigma d\xi_j \wedge dx_j$. Submanifolds of $T^*(X)$ of dimension *dim* X on which the symplectic form vanishes also play a fundamental role in the classical integration theory of first order differential equations (see section 3.1). Following Maslov [1] we shall call them *Lagrangean manifolds.*

Locally the class of distributions which can be written in the form (2.3.3) for some $a \in S_0^{m+n/4-N/2} (\Gamma)$, $n =$ *dim* X, and a non-degenerate real phase function φ depends only on the Lagrangean manifold Λ corresponding to φ and on no other properties of this function. Any closed conic Lagrangean submanifold $\Lambda \subset T^*(X)\backslash 0$ (or a closed conic subset of a Lagrangean submanifold which is not necessarily closed) can locally be represented as the range of a map (2.3.8). We can therefore define a space $I^m (X, \Lambda)$ of distributions with wave front set in Λ which locally can be written in the form (2.3.3) with $a \in S_0^{m+n/4-N/2}$ and φ defining a part of Λ according to (2.3.8). With the elements in $I^m (X, \Lambda)$ one can, as for pseudo-differential operators, associate principal symbols on Λ, which are symbols of order $m + n/4$ modulo symbols of order $m + n/4 - 1$ (with values in certain line bundles). The notion of characteristic point can therefore be defined as in section 2.1. For the kernels of pseudo-differential operators in X which are associated with the normal bundle of the diagonal in $X \times X$

this agrees with the standard notion of principal symbol. (Note that the normal bundle of any submanifold Y of X is Lagrangean in $T^*(X)$ and that the normal bundle of the diagonal in $X \times X$ can be identified with $T^*(X)$.)

When we take a conic Lagrangean submanifold of $T^*(X \times Y)\backslash 0$ where X and Y are two manifolds we can interpret the distributions in $I^m(X \times Y, \Lambda)$ as maps from $C_0^\infty(Y)$ to $\mathscr{D}'(X)$. When $\Lambda \subset \big(T^*(X)\backslash 0\big) \times \times \big(T^*(Y)\backslash 0\big)$ we have seen (Theorems 2.2.6 and 2.2.7) that they are actually continuous operators from $C_0^\infty(Y)$ to $C^\infty(X)$ and from $\mathscr{E}'(X)$ to $\mathscr{D}'(Y)$. The set

$$\Lambda' = \{(x, \xi, y, -\eta);\ (x, \xi, y, \eta) \in \Lambda\}$$

will then be called a homogeneous *canonical relation*; it is Lagrangean with respect to the symplectic form $\sigma_X - \sigma_Y$. This is the set which occurs in the multiplicative properties of wave front sets described in Theorem 2.2.8. If we have three manifolds X, Y, Z and canonical relations C_1, C_2 from $T^*(Y)$ to $T^*(X)$ resp. $T^*(Z)$ to $T^*(Y)$ one can supplement Theorem 2.2.8 by proving that the composition $K_1 \circ K_2$ of properly supported operators

$$K_1 \in I^{m_1}(X \times Y, C_1') \text{ and } K_2 \in I^{m_2}(Y \times Z, C_2')$$

is in

$$I^{m_1+m_2}(X \times Z, (C_1 \circ C_2)')$$

if the appropriate transversality and other conditions are fulfilled which guarantee that $C_1 \circ C_2$ is a manifold. There is a simple formula giving the principal symbol of $K_1 \circ K_2$ as a product of those of K_j. (The normalization of the degree for operators in I^m was chosen precisely to make the preceding statement valid.) For complete statements and proofs we refer to Hörmander [9]; a summary is given in Hörmander [10]. However, we shall consider an important special case due to Egorov [1] which gave rise to much of the work described here.

Thus assume that X and Y have the same dimension and that Λ' is the graph of a homogeneous *canonical transformation* χ from $T^*(Y)$ to $T^*(X)$ (or only a local canonical transformation in which case we consider a closed conic subset). That χ is canonical means that $\chi^*\sigma_X - \sigma_Y = 0$ or that $\sigma_X - \sigma_Y$ vanishes on Λ', so we have a canonical relation in the sense explained above. If $K \in I^m(X \times Y, \Lambda)$, then the adjoint K^* belongs to the inverse transformation and the compositions KK^* and K^*K belong to the identity, that is, they are pseudo-differential operators in X and in Y

respectively. If A is a pseudo-differential operator in X of order μ then the product AK is in $I^{m+\mu}(X \times Y, \Lambda)$ and the principal symbol is the product of the principal symbol of K (considered as living on Λ') by that of A lifted from $T^*(X)$ to Λ' by the projection $\Lambda' \to T^*(X)$. If we multiply to the right instead the result is the same except that we shall use the projection from Λ' to $T^*(Y)$. If A and B are pseudo-differential operators in X and in Y respectively and if $AK = KB$ we conclude that for the principal symbols a and b of A and B we must have

$$a(\chi(y, \eta)) = b(y, \eta) \tag{2.3.8}$$

if the principal symbol of K is not 0 at $(\chi(y, \eta), (y, -\eta))$. Conversely, (2.3.8) implies that $AK - KB$ is of lower order. We can therefore successively construct the symbol of B for a given A so that $AK - KB$ is of order $-\infty$, provided that the wave front set of A is concentrated near a point where K is elliptic. This argument often allows one to pass from one operator to another with principal symbol modified by a homogeneous canonical transformation. (See also Lemma 3.2.2 below.)

The operators in $I^m(X \times Y, \Lambda')$ can be described by means of the classical *generating function*: For any point $(x_0, \xi_0, y_0, \eta_0)$ in the graph of χ one can choose local coordinates in neighborhoods of x_0 and y_0 so that there is a function $\varphi(x, \eta)$ in a conical neighborhood of (x_0, η_0) which is homogeneous of degree 1 with respect to η, such that χ is given by $(\varphi'_\eta, \eta) \to \to (x, \varphi'_x)$ and $det\, \varphi''_{x\eta} \neq 0$. The elements A in $I^m(X \times Y, \Lambda)$ with wave front set close to $(x_0, \xi_0, y_0, -\eta_0)$ are then as operators of the form

$$Au(x) = (2\pi)^{-n} \int e^{i\varphi(x,\eta)} a(x, \eta)\, \hat{u}(\eta)\, d\eta, \quad a \in S^m(X \times \mathbf{R}^n),$$

when u is in C_0^∞ in a neighborhood of y_0 and x is in a neighborhood of x_0. The assertions made above are easy to prove directly from this representation.

Chapter III

Pseudo-differential operators with non-singular characteristics

3.1. *Preliminaries*

Throughout this chapter X will denote a C^∞ manifold (all manifolds are tacitly assumed countable at infinity) and P a properly supported pseudo-