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ON IDEAL-ADIC TOPOLOGIES FOR A COMMUTATIVE RING

by Robert GILMER

Let R be a commutative ring and let A and B be ideals of R such that $B \subseteq A$. We wish to consider the relationship between the following two conditions:

1) R is complete in the A-adic topology ¹).

2) R is complete in the B-adic topology.

Several results in this direction are known; for example:

THEOREM 1. ([4, Theorem 14, p. 275]) Assume that R is Noetherian with identity, and that R is complete Hausdorff in its A-adic topology. Then R is complete in its B-adic topology.

In [3], M. O'Malley proves the following theorem.

THEOREM 2. If R has an identity, and if R is complete Hausdorff in its A-adic topology, then R is complete in the (b)-adic topology for each element b of A.

In [2, Theorem 2.1 and Corollary 2.2], O'Malley extends his results in [3] to prove:

THEOREM 3. If R contains an identity, if $A = (a_1, ..., a_n)$ is finitely generated, and if R is Hausdorff in the A-adic topology, then R is complete in its A-adic topology if and only if R is complete in its (a_i) -adic topology for each i between 1 and n.

COROLLARY 1. If R contains an identity, and if R is a complete Hausdorff space in its A-adic topology, then R is complete Hausdorff in its B-adic topology for each finitely generated ideal B contained in A.

Moreover, O'Malley observes in [2] that Theorem 2, Theorem 3, and Corollary 1 are true without the assumption that R contains an identity, for the following result is valid.

¹) i.e. the topology for which a fundamental system of neighbourhoods of 0 is $A, A^2, A^3, ...$

PROPOSITION 1. Assume that R is a commutative ring, and S is a ring obtained by the canonical adjunction of an identity of characteristic zero to R (see [1, p. 4]). Let A be an ideal of R. Then A is an ideal of S and

1) R is Hausdorff in its A-adic topology if and only if S is Hausdorff in its A-adic topology;

2) R is complete in its A-adic topology if and only if S is complete in its A-adic topology.

O'Malley obtains the results we have cited from a much deeper theory of the set of *R*-endomorphisms of the power series ring R[[X]]. Our purpose here is to obtain O'Malley's results from basic topological considerations, independent of the theory of *R*-endomorphisms of R[[X]].

PROPOSITION 2. Assume that $\{A_i\}_{i=1}^n$ is a finite set of ideals of the commutative ring R, and let $A = A_1 + ... + A_n$. If R is complete in its A_i -adic topology for each i between 1 and n, then R is complete in its A-adic topology.

Proof. We note that the A-adic topology on R is the topology induced by the sequence $\{B_i\}_{i=1}^{\infty}$ of ideals, where $B_i = A_1^i + \ldots + A_n^i$. This is true because $A^i \supseteq B_i \supseteq A^{ni}$ for each positive integer *i*. Thus, if $\{c_i\}_0^{\infty}$ is a Cauchy sequence in the A-adic topology, then by passage to a subsequence of $\{c_i\}_0^{\infty}$, we can assume that $c_i - c_{i-1} \in B_i$ for each positive integer *i*. If we write $c_i - c_{i-1} = a_{1i} + a_{2i} + \ldots + a_{ni}$, where $a_{ji} \in A_j^i$, then for each *i*,

$$c_i = c_0 + \sum_{j=1}^n \sum_{k=1}^i a_{jk}$$
.

The series $\sum_{k=1}^{\infty} a_{jk}$ converges in the A_j -adic topology; we let $a_j^* = \lim_{k} (a_{j1} + a_{j2} + ... + a_{jk})$ Then it is clear that the sequence $\{c_i\}_{0}^{\infty}$ converges to $c_0 + \sum_{j=1}^{n} a_j^*$ in the A-adic topology. Therefore R is complete in its A-adic topology.

We remark that in Proposition 2, the A-adic topology on R need not be Hausdorff, although the A_i -adic topology is Hausdorff for each *i*. For example, if k is a field, then k [[X, Y, Z]]/A, where A = (Z(1-X-Y)), is complete Hausdorff under its [(X) + A]/A-adic and [(Y) + A]/A-adic topologies, but is not Hausdorff under its [(X, Y) + A]/A-adic topology.

THEOREM 4. Assume that R is a commutative ring, and that R is a complete Hausdorff space in its A-adic topology. If $b \in A$, then R is complete in its (b)-adic topology.

Proof. The (b)-adic topology on R is equivalent to the topology induced on R by the sequence $\{B_i\}_{i=1}^{\infty}$ of ideals, where $B_i = Rb^i$. This is true since $(b^i) \supseteq B_i \supseteq (b^{i+1})$ for each *i*. To prove that R is complete in its (b)-adic topology, it suffices to show that each sequence $\{c_i\}_{i=0}^{\infty}$, where $c_i - c_{i-1} \in B_i$ for each *i*, converges in the (b)-adic topology. Since $b \in A$, the sequence $\{c_i\}$ converges to an element c^* in the A-adic topology. We prove that c_i converges to c^* in the (b)-adic topology. Thus if $c_i - c_{i-1} = r_i b^i$ for each positive integer *i*, then for positive integers *k* and *n*, we have

$$c_{k+n} - c_k = b^{k+1} [r_{k+1} + r_{k+2}b + \dots + r_{k+n}b^{n-1}].$$

Taking limits in the A-adic topology as n approaches infinity, and using the fact that the A-adic topology is Hausdorff, we obtain

$$c^* - c_k = b^{k+1} s_{k+1}^*$$
 where $s_{k+1}^* = \sum_{n=1}^{\infty} r_{k+n} b^{n-1}$.

It follows that $c^* - c_k \in B_t$ for each $k \ge t - 1$, and $\{c_i\}$ converges to c^* in the (b)-adic topology, as asserted.

Theorem 4 fails if the assumption that R is Hausdorff in the A-adic topology is dropped. For example, if R is idempotent, then R is complete in its R-adic topology, but R need not be complete in its (b)-adic topology for each b in R. For a less trivial example, $Z \oplus Z$ is complete in its $(Z \oplus (0))$ -adic topology, but not in its $((2) \oplus (0))$ -adic topology.

Proposition 2 and Theorem 4 yield alternate proofs of Theorems 2 and 3 and Corollary 1 (dropping, in each case, the assumption that R has an identity).

We remark that in general, R need not be complete in its B-adic topology if R is complete Hausdorff in its A-adic topology, even if A is principal. Thus let D be an integral domain with identity containing a prime ideal $C = (c_1, c_2, ..., c_n, ...)$ such that C is countably generated, but C is not the radical of a finitely generated ideal. (For example, let $D = J[\{X_i\}_{i=1}^{\infty}]$, where J is an integral domain with identity and let $C = (\{X_i\}_{i=1}^{\infty}]$.) The ring R = D[[Y]] is a complete Hausdorff space in its (Y)-adic topology. But if $B = (\{cY \mid c \in C\})$, then R is not complete in the B-adic topology, for $\{c_1Y, c_1Y + c_2^2Y^2, ...\}$ is a Cauchy sequence in the B-adic topology which converges to $f = \sum_{i=1}^{\infty} c_i^i Y^i$ in the (Y)-adic topology. If this sequence converges in the B-adic topology, it must converge to f. But

$$f - \left(\sum_{1}^{n} c_{i}^{i} Y^{i}\right) = \sum_{n+1}^{\infty} c_{i}^{i} Y^{i} \notin B$$

for each positive integer, for if $\sum_{n+1}^{\infty} c_i^i Y^i \in B$, then for some positive integer k, $\sum_{n+1}^{\infty} c_i^i Y^i \in (c_1 Y, ..., c_k Y)$, and $c_i^i \in (c_1, ..., c_k)$ for each $i \ge n + 1$.

It follows that $C = \sqrt{(c_1, ..., c_k)}$, contrary to our assumptions concerning C.

Added in proof. Matthew O'Malley has pointed out to the author that in the remark preceding Theorem 4, the ring k[[X, Y, Z]]/A is Hausdorff in its [(X, Y) + A]/A-adic topology.

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