§1. The Hilbert modular group and the Euler number of its orbit space

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equivalent condition is that, for any compact subsets K_1, K_2 *of* X, *the set of all* $g \in G$ *with* $g(K_1) \cap K_2 \neq \emptyset$ *is finite.*

For a properly discontinuous action, the orbit space X/G is a Hausdorff space. For any $x \in X$, there exists a neighborhood U of x such that the (finite) set of all $g \in G$ with $gU \cap U \neq \emptyset$ equals the isotropy group $G_x = \{g \mid g \in G, g(x) = x\}$. If X is a normal complex space and G acts properly discontinuously by biholomorphic maps, then X/G is a normal complex space.

THEOREM. (H. Cartan [8], and [66] Exp. I). *If* X *is a bounded domain in* $\mathbf{C}^n$, *then the group* $\mathfrak{A}$ *of all biholomorphic maps* $X \to X$ *with the topology of compact convergence is a Lie group. For compact subsets* K_1, K_2 *of* X, *the set of all* $g \in \mathfrak{A}$ *such that* $gK_1 \cap K_2 \neq \emptyset$ *is a compact subset of* $\mathfrak{A}$. *A subgroup of* $\mathfrak{A}$ *is discrete if and only if it acts properly discontinuously.*

If X is a bounded symmetric domain, then a discrete subgroup Γ of $\mathfrak{A}$ operates freely if and only if it has no elements of finite order.

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§ 1. THE HILBERT MODULAR GROUP AND THE EULER NUMBER OF ITS ORBIT SPACE

1.1. Let $\mathfrak{H}$ be the upper half plane of all complex numbers with positive imaginary part. $\mathfrak{H}$ is embedded in the complex projective line $\mathbf{P}_1\mathbf{C}$. A complex matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc \neq 0$ operates on $\mathbf{P}_1\mathbf{C}$ by

$$z \mapsto \frac{az + b}{cz + d}$$

The matrices with real coefficients and $ad - bc > 0$ carry $\mathfrak{H}$ over into itself and constitute a group $\mathbf{GL}_2^+(\mathbf{R})$. The group

$$\mathbf{PL}_2^+(\mathbf{R}) = \mathbf{GL}_2^+(\mathbf{R}) / \{ \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) \mid a \neq 0 \} \tag{1}$$

operates effectively on $\mathfrak{H}$. As is well known, this is the group of all biholomorphic maps of $\mathfrak{H}$ to itself.

Writing $z = x + iy$ $(x, y \in \mathbf{R}, y > 0)$ we have on $\mathfrak{H}$ the Riemannian metric

$$\frac{(dx)^2 + (dy)^2}{y^2}$$

which is invariant under the action of $\mathbf{PL}_2^+(\mathbf{R})$. The volume element equals $y^{-2}\, dx \wedge dy$.

We introduce the Gauß-Bonnet form

$$\omega = -\frac{1}{2\pi} \cdot \frac{dx \wedge dy}{y^2} \tag{2}$$

If Γ is a discrete subgroup of $\mathbf{PL}_2^+(\mathbf{R})$ acting freely on $\mathfrak{H}$ and such that $\mathfrak{H}/\Gamma$ is compact, then $\mathfrak{H}/\Gamma$ is a compact Riemann surface of a certain genus p whose Euler number $e(\mathfrak{H}/\Gamma) = 2 - 2p$ is given by the formula

$$e(\mathfrak{H}/\Gamma) = \int_{\mathfrak{H}/\Gamma} \omega \tag{3}$$

We recall that the discrete subgroup Γ acts freely if and only if Γ has no elements of finite order.

1.2. Consider the n-fold cartesian product $\mathfrak{H}^n = \mathfrak{H} \times \ldots \times \mathfrak{H}$. Let $\mathfrak{A}$ be the group of all biholomorphic maps $\mathfrak{H}^n \to \mathfrak{H}^n$. The connectedness component of the identity of $\mathfrak{A}$ equals the n-fold direct product of $\mathbf{PL}_2^+(\mathbf{R})$ with itself. We have an exact sequence

$$1 \to \mathbf{PL}_2^+(\mathbf{R}) \times \ldots \times \mathbf{PL}_2^+(\mathbf{R}) \to \mathfrak{A} \to S_n \to 1, \tag{4}$$

where S_n is the group of permutations of n objects corresponding here to the permutations of the n factors of $\mathfrak{H}^n$. The sequence (4) presents $\mathfrak{A}$ as a semi-direct product. On $\mathfrak{H}^n$ we use coordinates $z_1, z_2, \ldots, z_n$ with $z_k = x_k + iy_k$ and $y_k > 0$. We have a metric invariant under $\mathfrak{A}$:

$$\sum_{j=1}^{n} \frac{(dx_j)^2 + (dy_j)^2}{y_j^2}$$

The corresponding Gauß-Bonnet form ω is obtained by multiplying the forms belonging to the individual factors; see (2). Therefore

$$\omega = (-1)^n \cdot \frac{1}{(2\pi)^n} \frac{dx_1 \wedge dy_1}{y_1{}^2} \wedge \ldots \wedge \frac{dx_n \wedge dy_n}{y_n{}^2} \tag{5}$$

If Γ is a discrete subgroup of $\mathfrak{A}$ acting freely on $\mathfrak{H}^n$ and such that $\mathfrak{H}^n/\Gamma$ is compact, then $\mathfrak{H}^n/\Gamma$ is a compact complex manifold whose Euler number is given by

$$e(\mathfrak{H}^n / \Gamma) = \int_{\mathfrak{H}^n/\Gamma} \omega. \tag{6}$$

$e(\mathfrak{H}^n/\Gamma)$ is always divisible by 2^n: for a compact complex n-dimensional manifold X we denote by $[X]$ the corresponding element in the complex cobordism group [58]. We have

$$[\mathfrak{H}^n / \Gamma] = 2^{-n} e(\mathfrak{H}^n / \Gamma) \cdot [(\mathbf{P}_1\mathbf{C})^n]. \tag{7}$$

This follows, because the Chern numbers of $\mathfrak{H}^n/\Gamma$ are proportional [37] to those of $(\mathbf{P}_1\mathbf{C})^n$. In particular, the Euler number and the arithmetic genus (Todd genus) of $(\mathbf{P}_1\mathbf{C})^n$ are 2^n and 1 respectively and thus $2^{-n} \cdot e(\mathfrak{H}^n/\Gamma)$ is the arithmetic genus of $\mathfrak{H}^n/\Gamma$.

1.3. We shall study special subgroups of the group of biholomorphic automorphisms of $\mathfrak{H}^n$. They are in fact discrete subgroups of $(\mathbf{PL}_2^+(\mathbf{R}))^n$. Let K be an algebraic number field of degree n over the field $\mathbf{Q}$ of rational numbers. We assume K to be totally real, i.e., there are n different embeddings of K into the reals. We denote them by

$$K \to \mathbf{R}, \; x \mapsto x^{(j)}, \; x \in K$$

We may assume $x = x^{(1)}$. The element x is called totally positive (in symbols, $x \gg 0$) if all $x^{(j)}$ are positive. The group

$$\mathbf{GL}_2^+(K) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid a, b, c, d \in K, ad - bc \gg 0 \}$$

acts on $\mathfrak{H}^n$ as follows: for $z = (z_1, \ldots, z_n) \in \mathfrak{H}^n$ we have

$$z_j \mapsto \frac{a^{(j)} z_j + b^{(j)}}{c^{(j)} z_j + d^{(j)}}.$$

The corresponding projective group

$$\mathbf{PL}_2^+(K) = \mathbf{GL}_2^+(K) / \{ \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right), a \in K^* \}$$

acts effectively on $\mathfrak{H}^n$. Thus $\mathbf{PL}_2^+(K) \subset (\mathbf{PL}_2^+(\mathbf{R}))^n$.

Let $\mathfrak{o}_K$ be the ring of algebraic integers in K, then by considering only matrices with $a, b, c, d \in \mathfrak{o}_K$ and $ad - bc = 1$ we get the subgroup $\mathbf{SL}_2(\mathfrak{o}_K)$ of $\mathbf{GL}_2^+(K)$. The group $\mathbf{SL}_2(\mathfrak{o}_K) / \{1, -1\}$ is the famous Hilbert modular group. It is a discrete subgroup of $(\mathbf{PL}_2^+(\mathbf{R}))^n$. We shall denote it by $G(K)$ or simply by G, if no confusion can arise.

$$G = \mathbf{SL}_2(\mathfrak{o}_K) / \{1, -1\} \subset \mathbf{PL}_2^+(K) \subset (\mathbf{PL}_2^+(\mathbf{R}))^n$$

The Hilbert modular group was studied by Blumenthal [5]. An error of Blumenthal concerning the number of cusps was corrected by Maaß [53].

The quotient space $\mathfrak{H}^n/G$ is not compact, but it has a finite volume with respect to the invariant metric. It is natural to use the Euler volume given in (5). The quotient space $\mathfrak{H}^n/G$ is a complex space and not a manifold (for $n > 1$). We shall return to this point later. But the volume of $\mathfrak{H}^n/G$ is well-defined and was calculated by Siegel ([72], [74]). The ζ-function of the field K enters. It is defined by

$$\zeta_K(s) = \sum_{\substack{\mathfrak{a} \subset \mathfrak{o}_K \\ \mathfrak{a} \text{ an ideal}}} \frac{1}{N(\mathfrak{a})^s}.$$

This sum extends over all ideals in $\mathfrak{o}_K$, and $N(\mathfrak{a})$ denotes the norm of $\mathfrak{a}$. The series converges if the real part of the complex number s is greater than 1. It converges absolutely uniformly on any compact set contained in the half plane $\mathrm{Re}(s) > 1$. The function ζ_K can be holomorphically extended to $\mathbf{C} - \{1\}$. It has a pole of order 1 for $s = 1$. Let D_K denote the discriminant of the field K.

Then

$$D_K^{\frac{s}{2}} \cdot \pi^{-\frac{sn}{2}} \cdot \Gamma(s/2)^n \cdot \zeta_K(s) \tag{8}$$

is invariant under the substitution $s \to 1 - s$.

This is the well-known functional equation of $\zeta_K(s)$. It can be found in most books on algebraic number theory. See, for example, [52].

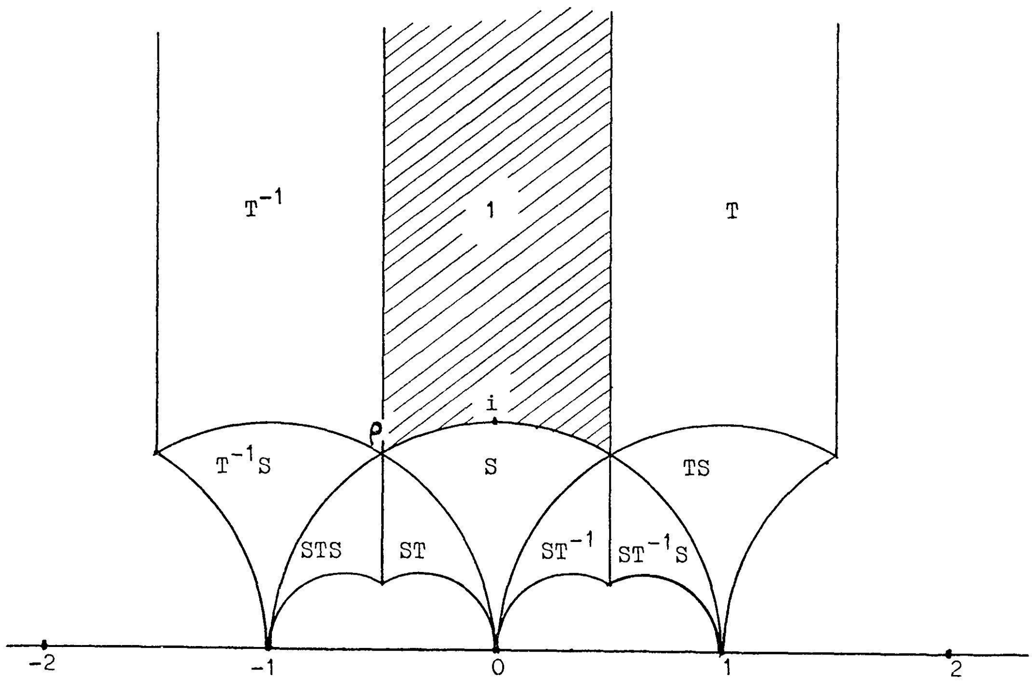
THEOREM (Siegel). *The Euler volume of $\mathfrak{H}^n/G$ relates to the zeta-function as follows*

(9) $$\int_{\mathfrak{H}^n/G} \omega = 2\zeta_K(-1).$$

The formula (19) of [72] uses the volume element $\dfrac{dx_1 \wedge dy_1}{{y_1}^2} \wedge \dots \wedge \dfrac{dx_n \wedge dy_n}{{y_n}^2}$ and gives for the volume the value $2\pi^{-n} \cdot {D_K}^{3/2} \zeta_K(2)$. If we multiply this value with $(-1)^n . (2\pi)^{-n}$, we get $\int_{\mathfrak{H}^n/G} \omega$.

Formula (9) follows from the functional equation. It was pointed out by J. P. Serre [69] that such Euler volume formulas may be written more conveniently using values of the zeta functions at negative odd integers. $2\zeta_K(-1)$ is a rational number, a result going back to Hecke, see Siegel ([73] Ges. Abh. I, p. 546, [76]) and Klingen [44]. The rational number $2\zeta_K(-1)$ is in fact the rational Euler number of G in the sense of Wall [77], as we shall see later.

1.4. We shall write down explicit formulas for $2\zeta_K(-1)$ in some cases. For $K = \mathbf{Q}$, the group G is the ordinary modular group acting on $\mathfrak{H}$. A fundamental domain is described by the famous picture (see, for example, [68] p. 128).



The volume of $\mathfrak{H}/G$ equals the volume of the shaded domain. By Siegel's general formula, the volume of the shaded domain with respect to $\dfrac{dx \wedge dy}{y^2}$ equals

$$2\pi^{-1}\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi}{3}\,.$$

Therefore, we get for the Euler volume

$$\int_{\mathfrak{H}/G}\omega=-\frac{1}{6}=2\zeta_{\mathbf{Q}}(-1)\,. \tag{10}$$

We consider the real quadratic fields $K=\mathbf{Q}(\sqrt{d})$ where d is a square-free natural number >1. We recall that the discriminant D of K is given by

$$D=4d \quad \text{for } d\equiv 2,3 \bmod 4$$
$$D=d \quad \text{for } d\equiv 1 \bmod 4.$$

The ring $\mathfrak{o}_K$ has additively the following $\mathbf{Z}$-bases.

$$\mathfrak{o}_K=\mathbf{Z}+\mathbf{Z}\sqrt{d} \qquad \text{for } d\equiv 2,3 \bmod 4$$
$$\mathfrak{o}_K=\mathbf{Z}+\mathbf{Z}\frac{1+\sqrt{d}}{2} \qquad \text{for } d\equiv 1 \bmod 4$$

THEOREM. *Let $K=\mathbf{Q}(\sqrt{d})$ be as above. Then for $d\equiv 1 \bmod 4$*

$$2\zeta_K(-1)=\frac{1}{15}\sum_{\substack{1\leq b<\sqrt{d}\\ b\ \text{odd}}}\sigma_1\left(\frac{d-b^2}{4}\right) \tag{11}$$

and for $d\equiv 2,3 \bmod 4$

$$2\zeta_K(-1)=\frac{1}{30}\Big(\sigma_1(d)+2\cdot\sum_{1\leq b<\sqrt{d}}\sigma_1(d-b^2)\Big) \tag{12}$$

where $\sigma_1(a)$ equals the sum of the divisors of a.

This theorem, though not exactly in this form, can be found in Siegel [76]. Compare also Gundlach [22], Zagier [78]. The κ_2 of Gundlach equals $4/\zeta_K(-1)$.

1.5. A reference for the following discussion is [71].

We always assume that Γ is a discrete subgroup of $(\mathbf{PL}^+(\mathbf{R}))^n$ and that $\mathfrak{H}^n/\Gamma$ has finite volume.

Γ is irreducible if it contains no element $\gamma = (\gamma^{(1)}, \ldots, \gamma^{(n)})$ such that $\gamma^{(i)} = 1$ for some i and $\gamma^{(j)} \neq 1$ for some j. See [71], p. 40 Corollary.

An element of $\mathbf{PL}_2^+(\mathbf{R})$ is parabolic if and only if it has exactly one fixed point in $\mathbf{P}_1\mathbf{C}$. This point belongs to $\mathbf{P}_1\mathbf{R} = \mathbf{R} \cup \infty$. An element $\gamma = (\gamma^{(1)}, \ldots, \gamma^{(n)})$ of $(\mathbf{PL}_2^+(\mathbf{R}))^n$ is called parabolic if and only if all $\gamma^{(i)}$ are parabolic. The parabolic element γ has exactly one fixed point in $(\mathbf{P}_1\mathbf{C})^n$. It belongs to $(\mathbf{P}_1\mathbf{R})^n$. The parabolic points of Γ are by definition fixed points of the parabolic elements of Γ.

The above notation, hopefully, will not confuse the reader. The $\gamma^{(i)}$ are simply the components of the element γ of $(\mathbf{PL}_2^+(\mathbf{R}))^n$. If $\gamma \in \mathbf{PL}_2^+(K) \subset (\mathbf{PL}_2^+(\mathbf{R}))^n$ (compare 1.3), then, for γ represented by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, the element $\gamma^{(i)}$ is represented by $\left(\begin{smallmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{smallmatrix}\right)$ where $x \mapsto x^{(i)}$ is the i-th embedding of K in $\mathbf{R}$. For any group $\Gamma \subset (\mathbf{PL}_2^+(\mathbf{R}))^n$ we consider the orbits of parabolic points under the action of Γ on $(\mathbf{P}_1\mathbf{R})^n$. They are called parabolic orbits. Each such orbit consists only of parabolic points.

If Γ is irreducible, then there are only finitely many parabolic orbits. ([71], p. 46 Theorem 5).

Hereafter we shall assume in addition that Γ is irreducible.

If $x \in (\mathbf{P}_1\mathbf{R})^n$ is a parabolic point of Γ, we transform it to $\infty = (\infty, \ldots, \infty)$ by an element ρ of $(\mathbf{PL}_2^+\mathbf{R})^n$, not necessarily belonging to Γ, of course. Thus $\rho x = \infty$.

Let Γ_x be the isotropy group of x.

$$\Gamma_x = \{\gamma \mid \gamma \in \Gamma, \gamma x = x\}.$$

Then any element of $\rho \Gamma_x \rho^{-1}$ is of the form

$$z_j \mapsto \lambda^{(j)} z_j + \mu^{(j)}, \lambda^{(j)} > 0. \tag{13}$$

Consider the following multiplicative group

$$\Lambda = \{t \mid t^{(i)} \in \mathbf{R}, t^{(i)} > 0, \prod_{j=1}^{n} t^{(j)} = 1\}. \tag{14}$$

It is isomorphic to $\mathbf{R}^{n-1}$ by taking logarithms. Each element of $\rho\Gamma_x\rho^{-1}$ (see (13)) satisfies $\lambda^{(1)} \cdot \lambda^{(2)} \ldots \cdot \lambda^{(n)} = 1$, (compare [71], p. 43, Theorem 3). Therefore we have a natural homomorphism $\rho\Gamma_x\rho^{-1} \to \Lambda$ whose image is a discrete subgroup Λ_x of Λ of rank $n-1$. The kernel consists of all the translations

$$z_j \mapsto z_j + \mu^{(j)}$$

where $\mu = (\mu^{(1)}, \ldots, \mu^{(n)})$ belongs to a certain discrete subgroup M_x of $\mathbf{R}^n$ of rank n. Thus we have an exact sequence

$$0 \to M_x \to \rho\,\Gamma_x \rho^{-1} \to \Lambda_x \to 1. \tag{15}$$

Using the inner automorphisms of $\rho\Gamma_x\rho^{-1}$, the group Λ_x acts on M_x by componentwise multiplication. However, in the general case, (15) does not present $\rho\Gamma_x\rho^{-1}$ as a semi-direct product. For $n = 1$, the group Λ_x is trivial. *For $n = 2$ it is infinite cyclic, $\rho\Gamma_x\rho^{-1}$ is a semi-direct product, and ρ can be chosen in such a way that $\rho\Gamma_x\rho^{-1}$ is exactly the group of all elements of the form* (13) *with $\lambda \in \Lambda_x$ and $\mu \in M_x$.*

For any positive number d, the group $\rho\Gamma_x\rho^{-1}$ acts freely on

$$W = \{ z \mid z \in \mathfrak{H}^n, \prod_{j=1}^{n} \operatorname{Im}(z_j) \geqq d \} \tag{16}$$

where Im denotes the imaginary part. The orbit space $W/\rho\Gamma_x\rho^{-1}$ is a (non-compact) manifold with compact boundary

$$N = \partial W / \rho\,\Gamma_x \rho^{-1}.$$

Since ∂W is a principal homogeneous space for the semi-direct product $E = \mathbf{R}^n \rtimes \Lambda$ of all transformations

$$z_j \mapsto t^{(j)} z_j + a^{(j)}, t \in \Lambda, a \in \mathbf{R}^n$$

we can consider N as the quotient space of the group E (homeomorphic to $\mathbf{R}^{2n-1}$) by the discrete subgroup $\rho\Gamma_x\rho^{-1}$. Thus N is an Eilenberg-MacLane space. The $(2n-1)$-dimensional manifold N is a torus bundle over the $(n-1)$-dimensional torus Λ/Λ_x. The fibre is the n-dimensional torus $\mathbf{R}^n/M_x$, and N is obtained by the action of Λ_x on $\mathbf{R}^n/M_x$ which is induced by the action $x_j \mapsto \lambda^{(j)} x_j + \mu^{(j)}$ of $\rho\Gamma_x\rho^{-1}$ on $\mathbf{R}^n$. Since, in general, $\mu^{(j)}$ is not necessarily an element of M_x, the action of Λ_x on $\mathbf{R}^n/M_x$ need not be the one given by componentwise multiplication.

Definition ([71], p. 48). *Let Γ be as before a discrete irreducible subgroup of $(\mathbf{PL}_2^+(\mathbf{R}))^n$ such that $\mathfrak{H}^n/\Gamma$ has finite volume. Let x_ν $(1 \leqq \nu \leqq t)$ be a complete set of Γ-inequivalent parabolic points of Γ. Choose elements $\rho_\nu \in (\mathbf{PL}_2^+(\mathbf{R}))^n$ with $\rho_\nu x_\nu = \infty$ and put $U_\nu = \rho_\nu^{-1}(W_\nu)$ where W_ν is defined as in* (16) *with some positive number d_ν instead of d. We say that Γ satisfies condition (F) if it admits (for some d_ν) a fundamental domain F of the form*

$$F = F_0 \cup V_1 \cup \ldots \cup V_t \qquad \text{(disjoint union)}$$

where F_0 *is relatively compact in* $\mathfrak{H}^n$ *and* V_ν *is a fundamental domain of* Γ_{x_ν} *in* U_ν.

The fundamental domain $F \subset \mathfrak{H}^n$ is by definition in one-to-one correspondence with $\mathfrak{H}^n/\Gamma$ and V_ν is in one-to-one correspondence with U_ν/Γ_{x_ν}.

The Hilbert modular group G of any totally real field K is a discrete irreducible subgroup of $(\mathbf{PL}_2^+(\mathbf{R}))^n$ with finite volume of $\mathfrak{H}^n/G$ which satisfies condition (F). The existence of a fundamental domain with the required properties was shown by Blumenthal [5] as corrected by Maaß [53]. See Siegel [75] for a detailed exposition.

Two subgroups of $(\mathbf{PL}_2^+(\mathbf{R}))^n$ are called commensurable if their intersection is of finite index in both of them.

Any subgroup Γ *of* $(\mathbf{PL}_2^+(\mathbf{R}))^n$ *which is commensurable with the Hilbert modular group* G *also satisfies* (F).

We define

$$[G:\Gamma] = [G:(G \cap \Gamma)]\,/\,[\Gamma:(G \cap \Gamma)] \tag{17}$$

Then we get for the Euler volume

$$\int_{\mathfrak{H}^n/\Gamma} \omega = [G:\Gamma] \cdot \int_{\mathfrak{H}^n/\Gamma} \omega = [G:\Gamma] \cdot 2\zeta_K(-1) \tag{18}$$

Remark. It is not known whether every discrete irreducible subgroup Γ of $(\mathbf{PL}_2^+(\mathbf{R}))^n$ such that $\mathfrak{H}^n/\Gamma$ has finite volume satisfies Shimizu's condition (F).

Selberg has conjectured that any Γ satisfying (F) and having at least one parabolic point $(t \geqq 1)$ is conjugate in the group $\mathfrak{A}$ of all automorphism of $\mathfrak{H}^n$ to a group commensurable with the Hilbert modular group G of some totally real field K with $[K:\mathbf{Q}] = n$.

1.6. Harder [28] has proved a general theorem on the Euler number of not necessarily compact quotient spaces of finite volume. For the following result a direct proof can be given by the method used in [40].

THEOREM (Harder). *Let* $\Gamma \subset (\mathbf{PL}_2^+(\mathbf{R}))^n$ *be a discrete irreducible group satisfying condition* (F) *of the definition in* 1.5. *Suppose moreover* *that Γ operates freely on $\mathfrak{H}^n$. Then $\mathfrak{H}^n/\Gamma$ is a complex manifold whose Euler number is given by*

$$e(\mathfrak{H}^n/\Gamma) = \int_{\mathfrak{H}^n/\Gamma} \omega. \tag{19}$$

If Γ is commensurable with the Hilbert modular group G of K, (where K is a totally real field of degree n over **Q***) then*

$$e(\mathfrak{H}^n/\Gamma) = [G:\Gamma]\cdot 2\zeta_K(-1). \tag{20}$$

Proof. It follows from 1.5 that $\mathfrak{H}^n/\Gamma$ contains a compact manifold Y with t boundary components $B_\nu = \partial W_\nu/\rho\Gamma_x\rho^{-1}$ (which are T^n-bundles over T^{n-1}). We have to choose the numbers d_ν sufficiently large. By the Gauß-Bonnet theorem of Allendoerfer-Weil-Chern [10]

$$e(\mathfrak{H}^n/\Gamma) = \int_Y \omega + \sum_{\nu=1}^{t} \int_{B_\nu} \prod$$

where $\prod$ is a certain $(2n-1)$-form. By the argument explained in [40], one can show easily that

$$\lim_{d_\nu\to\infty} \int_{B_\nu} \prod = 0. \text{ Q.E.D.}$$

Since the Hilbert modular group G always contains a subgroup Γ of finite index which operates freely and since $\mathfrak{H}^n/\Gamma$ can be replaced up to homotopy by the compact manifold Y with boundary, $[G:\Gamma]\cdot 2\zeta_K(-1)$ is the Euler number of Γ in the sense of the rational cohomology theory of groups and thus $2\zeta_K(-1)$ is the Euler number of G in the sense of Wall [77].

THEOREM. *Let $\Gamma \subset (\mathbf{PL}_2^+(\mathbf{R}))^n$ be a discrete irreducible group such that $\mathfrak{H}^n/\Gamma$ has finite volume. Assume that Γ satisfies condition (F). The isotropy groups Γ_z ($z\in\mathfrak{H}^n$) are finite cyclic and the set of those z with $|\Gamma_z| > 1$ projects down to a finite set in $\mathfrak{H}^n/\Gamma$. Thus $\mathfrak{H}^n/\Gamma$ is a complex space with finitely many singularities.* (For $n = 1$, these "branching points" are actually not singularities.)

Let $a_r(\Gamma)$ be the number of points in $\mathfrak{H}^n/\Gamma$ which come from isotropy groups of order r. The Euler number of the space $\mathfrak{H}^n/\Gamma$ is well-defined, and we have

$$(21) \qquad e(\mathfrak{H}^n/\Gamma) = \int_{\mathfrak{H}^n/\Gamma} \omega + \sum_{r \geqq 2} a_r(\Gamma) \frac{r-1}{r}.$$

The proof is an easy consequence of the Allendoerfer-Weil-Chern formula (compare [40], [65]).

The easiest example of (21) is of course the ordinary modular group $G = G(\mathbf{Q})$. We have $a_2(G) = a_3(G) = 1$ whereas the other $a_r(G)$ vanish. Thus

$$e(\mathfrak{H}/G) = -\tfrac{1}{6} + \tfrac{1}{2} + \tfrac{2}{3} = 1.$$

This checks, since $\mathfrak{H}/G$ and $\mathbf{C}$ are biholomorphically equivalent.

1.7. We shall apply (21) to the Hilbert modular group G and the extended Hilbert modular $\hat{G}$ of a real quadratic field. $\hat{G}$ is defined for any totally real field K. To define it we must say a few words about the units of K. They are the units of the ring $\mathfrak{o}_K$ of algebraic integers. Let U be the group of these units. Its rank equals $n - 1$ by Dirichlet's theorem [6]. Let U^+ be the group of all totally positive units (see 1.3). It also has rank $n - 1$ because it contains $U^2 = \{\varepsilon^2 \mid \varepsilon \in U\}$.

The extended Hilbert modular group is defined as follows

$$\hat{G} = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid a, b, c, d \in \mathfrak{o}_K, ad - bc \in U^+ \} / \{ \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) \mid a \in U \}$$

We have an exact sequence

$$1 \to G \to \hat{G} \to U^+ / U^2 \to 1.$$

obtained by associating to each element of $\hat{G}$ its determinant mod U^2.

If $K = \mathbf{Q}(\sqrt{d})$ with d as in 1.4., then U^+ and U^2 are infinite cyclic groups and U^+/U^2 is of order 2 or 1. The first case happens if and only if there is no unit in $\mathfrak{o}_K$ with negative norm. If d is a prime p, then

$$U^+ \neq U^2 \Leftrightarrow p \equiv 3 \bmod 4$$
$$U^+ = U^2 \Leftrightarrow p = 2 \text{ or } p \equiv 1 \bmod 4.$$

Compare [30], Satz 133.

To apply (21) to the groups G and $\hat{G}$ belonging to a real quadratic field we must know the numbers $a_r(G)$ and $a_r(\hat{G})$. They were determined by Gundlach [21] in some cases and in general by Prestel [61] using the idea that the isotropy groups G_z and $\hat{G}_z$ respectively ($z \in \mathfrak{H}^2$) determine orders in imaginary extensions of K, which by an additional step relates the $a_r(G)$ and $a_r(\hat{G})$ to ideal class numbers of quadratic imaginary fields over $\mathbf{Q}$. To write down Prestel's result we fix the following notation. A quadratic field k over $\mathbf{Q}$ (real or imaginary) is completely given by its discriminant D. The class number of the field will be denoted by $h(D)$ or by $h(k)$.

Prestel has very explicit results for the Hilbert modular group G of any real quadratic field K and for the extended group $\hat{G}$ in case the class number of K is odd. We shall indicate part of his result.

THEOREM. (Prestel). *Let d be squarefree, $d \geqq 7$ and $(d, 6) = 1$. Let $K = \mathbf{Q}(\sqrt{d})$. Then for the Hilbert modular group $G(K)$ we have for*

$d \equiv 1 \bmod 4$

$$a_2(G) = h(-4d), a_3(G) = h(-3d), a_r(G) = 0 \text{ for } r \neq 2, 3$$

and for $d \equiv 3 \bmod 8$

$$a_2(G) = 10 \cdot h(-d), a_3(G) = h(-12d), a_r(G) = 0 \text{ for } r \neq 2, 3$$

and for $d \equiv 7 \bmod 8$

$$a_2(G) = 4h(-d), a_3(G) = h(-12d), a_r(G) = 0 \text{ for } r \neq 2, 3$$

If d is a prime $\equiv 3 \bmod 4$ *and* $d \neq 3$ *we have for the extended group $\hat{G}(K)$ the following result:*

If $d \equiv 3 \bmod 8$, *then*

$$a_2(\hat{G}) = 3h(-d) + h(-8d), a_3(\hat{G}) = h(-12d)/2,$$
$$a_4(\hat{G}) = 4h(-d),$$
$$a_r(\hat{G}) = 0 \text{ for } r \neq 2, 3, 4.$$

If $d \equiv 7 \bmod 8$, *then*

$$a_2(\hat{G}) = h(-d) + h(-8d), a_3(\hat{G}) = h(-12d)/2,$$
$$a_4(\hat{G}) = 2h(-d),$$
$$a_r(\hat{G}) = 0 \text{ for } r \neq 2, 3, 4.$$

Prestel gives the numbers $a_r(G)$ and $a_r(\hat{G})$ also for $d = 2, 3, 5$. For $d = 3$ we have

$$a_2(\hat{G}) = 3,\ a_3(\hat{G}) = 1,\ a_4(\hat{G}) = 1,\ a_{12}(\hat{G}) = 1,$$

all other $a_r(\hat{G}) = 0$.

We apply (12), (20) and (21) for $K = \mathbf{Q}(\sqrt{3})$ as an example

$$2\zeta_K(-1) = \frac{1}{30}(4 + 2\sigma_1(2)) = \frac{10}{30} = \frac{1}{3},$$

$$[G : \hat{G}] = \frac{1}{2},$$

$$e(H^2/\hat{G}) = \frac{1}{6} + 3 \cdot \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{11}{12} = 4.$$

We shall copy Prestel's table [61] of the $a_r(G)$ and the $a_r(\hat{G})$ (if known) for $K = \mathbf{Q}(\sqrt{d})$ up to $d = 41$. In [61] the table contains an error which was corrected in [62].

We also tabulate the values of $2\zeta_K(-1)$, $e(\mathfrak{H}^2/G)$, and of $e(\mathfrak{H}^2/\hat{G})$ if known. In the columns before $2\zeta_K(-1)$ we find the values of the $a_r(G)$; the values of the $a_r(\hat{G})$ are written behind $2\zeta_K(-1)$. If there is no entry, then the value is zero.

If the $a_r(\hat{G})$ and $e(\mathfrak{H}^2/\hat{G})$ are not given in the table, this means that either there exists a unit of negative norm and thus $G = \hat{G}$ or that the values are not known. This is indicated in the last column.

By Prestel $a_r(G) = 0$ for $r > 3$ and $K = \mathbf{Q}(\sqrt{d})$ with $d > 5$, and we have for $d > 5$

(22) $$e(\mathfrak{H}^n/G) = 2\zeta_K(-1) + \frac{a_2(G)}{2} + a_3(G) \cdot \frac{2}{3}$$

Since the Euler number is an integer, we obtain by (11) and (12):

For $d > 5$, $d \equiv 1 \bmod 4$, d square-free,

$$\sum_{\substack{1 \leq b < \sqrt{d} \\ b \text{ odd}}} \sigma_1\left(\frac{d - b^2}{4}\right) \equiv 0 \bmod 5$$

For $d > 5$, $d \equiv 2, 3 \bmod 4$, d square-free

$$\sigma_1(d) + 2 \sum_{1 \leq b < \sqrt{d}} \sigma_1(d - b^2) \equiv 0 \bmod 5$$

Problem. Prove these congruences in the framework of elementary number theory.

d	2	3	4	5	6	$2\zeta_K(-1)$	2	3	4	6	12	$e(\mathfrak{H}^2/G)$	$e(\mathfrak{H}^2/\hat{G})$
2	2	2	2			1/6	—	—	—	—	—	4	$G = \hat{G}$
3	3	2			1	1/3	3	1	1		1	4	4
5	2	2		2		1/15	—	—	—	—	—	4	$G = \hat{G}$
6	6	3				1	5	1	2	1		6	6
7	4	4				4/3	5	2	2			6	6
10	6	4				7/3	—	—	—	—	—	8	$G = \hat{G}$
11	10	4				7/3	5	2	4			10	8
13	2	4				1/3	—	—	—	—	—	4	$G = \hat{G}$
14	12	4				10/3	8	2	4			12	10
15	8	6				4	—	—	—	—	—	12	?
17	4	2				2/3	—	—	—	—	—	4	$G = \hat{G}$
19	10	4				19/3	9	2	4			14	12
21	4	5				2/3	3	2		1		6	4
22	6	8				23/3	12	4	2			16	14
23	12	8				20/3	7	4	6			18	14
26	18	4				25/3	—	—	—	—	—	20	$G = \hat{G}$
29	6	6				1	—	—	—	—	—	8	$G = \hat{G}$
30	12	10				34/3	—	—	—	—	—	24	?
31	12	4				40/3	11	2	6			22	18
33	4	3				2	7	1		1		6	6
34	12	4				46/3	—	—	—	—	—	24	?
35	20	8				38/3	—	—	—	—	—	28	?
37	2	8				5/3	—	—	—	—	—	8	$G = \hat{G}$
38	18	8				41/3	16	4	6			28	22
39	16	10				52/3	—	—	—	—	—	40	?
41	8	2				8/3	—	—	—	—	—	8	$G = \hat{G}$

§ 2. The cusps and their resolution for the 2-dimensional case

2.1. Let K be a totally real algebraic field of degree n over $\mathbf{Q}$ and M an additive subgroup of K which is a free abelian group of rank n. Such a group M is called a complete $\mathbf{Z}$-module of K. Let U_M^+ be the group of those units ε of K which are totally positive and satisfy $\varepsilon M = M$. Any $\alpha \in K$ with $\alpha M = M$ is automatically an algebraic integer and a unit.

The group U_M^+ is free of rank $n - 1$ (compare [6]).