§4. Curves on the Hilbert modular surfaces AND PROOFS OF RATIONALITY

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THEOREM. *Let p be a prime* $K = \mathbf{Q}(\sqrt{p})$. *Let G be the Hilbert modular group for K and* $\hat{G}$ *the extended one. Then*

$$\chi(G) = 1 \qquad \text{for } p = 2, 3, 5$$
$$\chi(\hat{G}) = 1 \qquad \text{for } p = 3$$

For $p > 5$ we have

$$\chi(G) = \frac{1}{2}\zeta_K(-1) + \frac{h(-4p)}{8} + \frac{1}{6}h(-3p) \qquad \text{for } p \equiv 1 \bmod 4$$

$$\chi(G) = \frac{1}{2}\zeta_K(-1) + \frac{3}{4}h(-p) + \frac{1}{6}h(-12p) \qquad \text{for } p \equiv 3 \bmod 8$$

$$\chi(G) = \frac{1}{2}\zeta_K(-1) + \frac{1}{6}h(-12p) \qquad \text{for } p \equiv 7 \bmod 8$$

$$\chi(\hat{G}) = \frac{1}{4}\zeta_K(-1) + \frac{9}{8}h(-p) + \frac{1}{8}h(-8p) + \frac{1}{12}h(-12p) \qquad \text{for } p \equiv 3 \bmod 8$$

$$\chi(\hat{G}) = \frac{1}{4}\zeta_K(-1) + \frac{1}{8}h(-8p) + \frac{1}{12}h(-12p) \qquad \text{for } p \equiv 7 \bmod 8$$

The formulas at the end of 1.3 imply

$$2\zeta_K(-1) = \frac{1}{2}\cdot\pi^{-4}D_K^{3/2}\zeta_K(2) > \frac{1}{2}\pi^{-4}D_K^{3/2}\zeta(4) = \frac{D_K^{3/2}}{180}.$$

It is easy to deduce from this estimate that $\chi(G) = 1$ if and only if $p = 2, 3, 5, 7, 13, 17$ and (for $p \equiv 3 \bmod 4$) $\chi(\hat{G}) = 1$ if and only if $p = 3, 7$. Because of (38) and (56) we also know the arithmetic genera of $\overline{(\mathfrak{H} \times \mathfrak{H}^-)/G}$ and $\overline{(\mathfrak{H} \times \mathfrak{H}^-)/\hat{G}}$ ($p \equiv 3 \bmod 4$). They are equal to 1 if $p = 3$, and both different from 1 if $p > 3$.

§ 4. CURVES ON THE HILBERT MODULAR SURFACES AND PROOFS OF RATIONALITY

We shall construct curves in the Hilbert modular surfaces. They can be used to show that these surfaces are rational in some cases and also for further investigations of the surfaces ([41], [42]). Such curves were studied earlier by Gundlach [23] and Hammond [25]. We need information about the decomposition of numbers into prime ideals in quadratic fields. (See [6], [30].)

4.1. Let K be a real quadratic field and $\mathfrak{o}_K$ its ring of integers. We often write $\mathfrak{o}$ instead of $\mathfrak{o}_K$. *Let* $\mathfrak{b}$ *be an ideal in* $\mathfrak{o}$ *which is not divisible by any natural number* > 1. We consider the group $\mathbf{SL}_2(\mathfrak{o}, \mathfrak{b})$, see § 3 (41). Let $\Gamma_\mathfrak{b}$ be the subgroup of those elements of $\mathbf{SL}_2(\mathfrak{o}, \mathfrak{b})$ which when acting on $\mathfrak{H}^2$ carry the diagonal $z_1 = z_2$ over into itself. An element $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ of $\mathbf{SL}_2(\mathfrak{o}, \mathfrak{b})$ belongs to $\Gamma_\mathfrak{b}$ if and only if

(1) $$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}-\alpha' & -\beta'\\ -\gamma' & -\delta'\end{pmatrix}$$

The matrices satisfying the first condition of (1) are in $\mathbf{SL}_2(\mathbf{Q})$ with $\alpha, \delta \in \mathfrak{o}$, $\beta \in \mathfrak{b}^{-1}$, $\gamma \in \mathfrak{b}$. Thus α, δ, γ are integers. The ideal $\mathfrak{b}$ is not divisible by any natural number > 1. Therefore β is also an integer. A rational integer γ is contained in $\mathfrak{b}$ if and only if $\gamma \equiv 0 \bmod N(\mathfrak{b})$ where $N(\mathfrak{b})$ is the norm of the ideal $\mathfrak{b}$.

For any natural number N we let $\Gamma_0(N)$ be the group of those elements $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathbf{SL}_2(\mathbf{Z})$ for which $\gamma \equiv 0 \bmod N$. This group was studied by Klein and Fricke ([16], p. 349; see [70], p. 24).

We have proved the following lemma:

Lemma. Let $\mathfrak{b}$ *be an ideal in* $\mathfrak{o}$ *which is not divisible by any natural number* > 1. *Then* $\Gamma_0(N(\mathfrak{b}))$ *is the subgroup of those elements of* $\Gamma_\mathfrak{b}$ *which satisfy the first condition of* (1). *The group* $\Gamma_\mathfrak{b}$ *equals* $\Gamma_0(N(\mathfrak{b}))$ *or is an extension of index* 2 *of* $\Gamma_0(N(\mathfrak{b}))$.

If $K = \mathbf{Q}(\sqrt{d})$ where d is square free, then a matrix of $\Gamma_\mathfrak{b}$ satisfying the second condition of (1) is of the form $\sqrt{d}\left(\begin{smallmatrix}\alpha_0 & \beta_0\\ \gamma_0 & \delta_0\end{smallmatrix}\right)$ where $\alpha_0, \gamma_0, \delta_0$ are rational integers, β_0 is a rational number, $\gamma_0\sqrt{d} \in \mathfrak{b}$, and $\beta_0\sqrt{d} \in \mathfrak{b}^{-1}$.

If $\mathfrak{b}$ is not divisible by $(\sqrt{d})$, then the fractional ideal $(\beta_0\sqrt{d})$ has in its numerator a prime ideal dividing the ideal $(\sqrt{d})$ and the determinant of our matrix would be divisible by this prime ideal, this is a contradiction. Thus a matrix satisfying the second condition of (1) does not exist in this case. If $\mathfrak{b}$ is divisible by $(\sqrt{d})$, then $[\Gamma_\mathfrak{b} : \Gamma_0(N(b))] = 2$.

In fact, $\mathfrak{b}$ is divisible by $(\sqrt{d})$ if and only if $N(\mathfrak{b})$ is divisible by d and the matrices satisfying the second condition of (1) are of the form

$$\begin{pmatrix} \alpha_0 \sqrt{d} & \beta_1 / \sqrt{d} \\ \gamma_0 \sqrt{d} & \delta_0 \quad \sqrt{d} \end{pmatrix}$$

where $\alpha_0, \beta_1, \gamma_0, \delta_0$ are rational integers and $\gamma_0 \equiv 0 \bmod N(\mathfrak{b})/d$. Such matrices exist, because $(d, N(\mathfrak{b})/d) = 1$. If $\mathfrak{b} = (\sqrt{d})$, then $\Gamma_{\mathfrak{b}}$ is the extension of index 2 of $\Gamma_0(d)$ by the matrix

$$\begin{pmatrix} 0 & -1/\sqrt{d} \\ \sqrt{d} & 0 \end{pmatrix}$$

This group will be denoted by $\Gamma^*(d)$, see Fricke ([16], p. 357). We have proved:

Proposition: Let $K = \mathbf{Q}(\sqrt{d})$ *be a real quadratic field (d square free). Let* $\mathfrak{b}$ *be an ideal in* $\mathfrak{o}_K$ *which is not divisible by any natural number* > 1. *If* $N(\mathfrak{b})$ *is not divisible by d, then the group* $\Gamma_{\mathfrak{b}}$ *of those elements of* $\mathbf{SL}_2(\mathfrak{o}_K, \mathfrak{b})$ *which carry the diagonal of* $\mathfrak{H}^2$ *into itself equals* $\Gamma_0(N(\mathfrak{b}))$. *If* $N(\mathfrak{b})$ *is divisible by d, then* $\Gamma_{\mathfrak{b}}$ *is an extension of index* 2 *of* $\Gamma_0(N(\mathfrak{b}))$. *In particular, if* $\mathfrak{b} = (\sqrt{d})$, *then* $\Gamma_{\mathfrak{b}} = \Gamma^*(d)$.

We also consider the group $\widehat{\mathbf{SL}}_2(\mathfrak{o}_K, \mathfrak{b})$ of matrices $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ with $\alpha, \delta \in \mathfrak{o}_K$, $\beta \in \mathfrak{b}^{-1}$, $\gamma \in \mathfrak{b}$ and $\alpha\delta - \beta\gamma$ a totally positive unit.

The groups $\mathbf{SL}_2(\mathfrak{o}_K, \mathfrak{b})$ and $\widehat{\mathbf{SL}}_2(\mathfrak{o}_K, \mathfrak{b})$ do not act effectively on $\mathfrak{H}^2$. If we divide them by their subgroups of diagonal matrices, we get the groups $G(\mathfrak{o}_K, \mathfrak{b})$ and $\hat{G}(\mathfrak{o}_K, \mathfrak{b})$ which act effectively and generalize the Hilbert modular groups G and $\hat{G}$ (see 1.7). As in 1.7 we have an exact sequence

$$(2) \qquad 0 \to G(\mathfrak{o}_K, \mathfrak{b}) \to \hat{G}(\mathfrak{o}_K, \mathfrak{b}) \to U^+/U^2 \to 0$$

The subgroup of those elements of $G(\mathfrak{o}_K, \mathfrak{b})$ which carry the diagonal over into itself is $\Gamma_{\mathfrak{b}}/\{1, -1\}$ which acts effectively on $\mathfrak{H}$. The subgroup of the elements of $\hat{G}(\mathfrak{o}_K, \mathfrak{b})$ which keep the diagonal invariant is an extension of index 1 or 2 of $\Gamma_{\mathfrak{b}}/\{1, -1\}$. We can write it in the form $\hat{\Gamma}_{\mathfrak{b}}/\{1, -1\}$ where $\hat{\Gamma}_{\mathfrak{b}} \subset \mathbf{SL}_2(\mathbf{R})$ is an extension of index 1 or 2 of $\Gamma_{\mathfrak{b}}$.

The embedding of the diagonal in $\mathfrak{H}^2$ induces maps π and $\hat{\pi}$ of $\mathfrak{H}/\Gamma_{\mathfrak{b}}$ and $\mathfrak{H}/\hat{\Gamma}_{\mathfrak{b}}$ in $\mathfrak{H}^2/G(\mathfrak{o}_K, \mathfrak{b})$ and $\mathfrak{H}^2/\hat{G}(\mathfrak{o}_K, \mathfrak{b})$ respectively. The maps π and $\hat{\pi}$ need not be injective. We have a commutative diagram

$$
(3) \qquad
\begin{array}{ccc}
\mathfrak{H}/\Gamma_{\mathfrak{b}} & \xrightarrow{\quad \pi \quad} & \mathfrak{H}^2/G(\mathfrak{o}_K, \mathfrak{b}) \\
\sigma \big\downarrow & & \big\downarrow \rho \\
\mathfrak{H}/\hat{\Gamma}_{\mathfrak{b}} & \xrightarrow{\quad \hat{\pi} \quad} & \mathfrak{H}^2/\hat{G}(\mathfrak{o}_K, \mathfrak{b})
\end{array}
$$

The maps π and $\hat{\pi}$ map $\mathfrak{H}/\Gamma_{\mathfrak{b}}$ and $\mathfrak{H}/\hat{\Gamma}_{\mathfrak{b}}$ with degree 1 onto their images.

If K has a unit of negative norm, then the two lines of diagram (3) can be identified. If there does not exist a unit of negative norm in K, then ρ has degree 2 and σ is bijective or has degree 2, depending on whether

$$\hat{\Gamma}_{\mathfrak{b}} = \Gamma_{\mathfrak{b}} \quad \text{or} \quad [\hat{\Gamma}_{\mathfrak{b}} : \Gamma_{\mathfrak{b}}] = 2.$$

If we compactify $\mathfrak{H}^2/G(\mathfrak{o}_K, \mathfrak{b})$ and $\mathfrak{H}^2/\hat{G}(\mathfrak{o}_K, \mathfrak{b})$ and resolve all quotient and cusp singularities by their minimal resolutions, then we get non-singular algebraic surfaces $Y(\mathfrak{o}_K, \mathfrak{b})$ and $\hat{Y}(\mathfrak{o}_K, \mathfrak{b})$. On $Y(\mathfrak{o}_K, \mathfrak{b})$ we have an involution α induced by $\left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & 1 \end{smallmatrix}\right)$ where ε is a generator of U^+. We have a rational map $\rho : Y(\mathfrak{o}_K, \mathfrak{b}) \to \hat{Y}(\mathfrak{o}_K, \mathfrak{b})$ compatible with α. The map ρ is regular outside the isolated fixed points of α. The maps π and $\hat{\pi}$ induce maps of the compactifications $\overline{\mathfrak{H}/\Gamma_{\mathfrak{b}}}$ and $\overline{\mathfrak{H}/\hat{\Gamma}_{\mathfrak{b}}}$ into the non-singular algebraic surfaces. We have a commutative diagram

$$
(4) \qquad
\begin{array}{ccc}
\overline{\mathfrak{H}/\Gamma_{\mathfrak{b}}} & \xrightarrow{\quad \pi \quad} & Y(\mathfrak{o}_K, \mathfrak{b}) \\
\sigma \big\downarrow & & \big\downarrow \rho \\
\overline{\mathfrak{H}/\hat{\Gamma}_{\mathfrak{b}}} & \xrightarrow{\quad \hat{\pi} \quad} & \hat{Y}(\mathfrak{o}_K, \mathfrak{b})
\end{array}
$$

If K has a unit of negative norm, then the two lines of (4) can be identified, the vertical maps are bijective.

We denote the irreducible curve $\hat{\pi}(\overline{\mathfrak{H}/\hat{\Gamma}_{\mathfrak{b}}})$ by $C(\mathfrak{b})$. It may have singularities. $\overline{\mathfrak{H}/\hat{\Gamma}_{\mathfrak{b}}}$ is its non-singular model which is mapped by $\hat{\pi}$ with degree 1 *on $C(\mathfrak{b})$.*

We put $D(\mathfrak{b}) = \rho^{-1} C(\mathfrak{b})$. If degree $(\rho) = 2$ and degree $(\sigma) = 1$ then $D(\mathfrak{b})$ is the union of two irreducible curves $D_1(\mathfrak{b})$, $D_2(\mathfrak{b})$. If degree $(\sigma) = 2$, then $D(\mathfrak{b})$ is irreducible. The involution α carries $D(\mathfrak{b})$ into itself, mapping $D_1(\mathfrak{b})$ to $D_2(\mathfrak{b})$ if $D(\mathfrak{b})$ is reducible.

The resolution of the cusp at ∞ of $\overline{\mathfrak{H}^2/\hat{G}(\mathfrak{o}_K, \mathfrak{b})}$ is described by the primitive cycle $((b_0, \ldots, b_{r-1}))$ of $\mathfrak{b}^{-1}$ (see 2.6) It determines a (narrow) ideal class with respect to strict equivalence whose inverse we denote by $\mathfrak{B}$. A quadratic irrationality w is called reduced if $0 < w' < 1 < w$. The quadratic irrationality w is reduced if and only if its continued fraction is purely periodic. There are exactly r reduced quadratic irrationalities belonging to the cycle, namely the numbers

$$w_k = [[b_k, b_{k+1}, \ldots]], \quad \text{(see 2.3 (8))}. \tag{5}$$

After calling one of them w_0, the notation for the others is fixed. Then they correspond bijectively to $\mathbf{Z}/r\mathbf{Z}$.

If we speak of the curve S_k of the resolution (where $k \in \mathbf{Z}/r\mathbf{Z}$), this has an invariant meaning. It is the curve associated to the quadratic irrationality w_k.

The fractional ideals $\mathfrak{b}^{-1} \in \mathfrak{B}^{-1}$ (where $\mathfrak{b} \subset \mathfrak{o}_K$ and $\mathfrak{b}$ is not divisible by any natural number > 1) are exactly the $\mathbf{Z}$-modules $\mathbf{Z}w + \mathbf{Z} \cdot 1$ where w is a quadratic irrationality having the given primitive cycle in its continued fraction. (If we require that $0 < w' < 1$ and $w' < w$, then w is uniquely determined by $\mathfrak{b}^{-1}$.)

Since the module $\mathfrak{b}^{-1} = \mathbf{Z}w + \mathbf{Z} \cdot 1$ is strictly equivalent to $M = \mathbf{Z}w_0 + \mathbf{Z} \cdot 1$ (see 2.3), there exists a totally positive number λ in M (uniquely determined up to multiplication by a totally positive unit) such that

$$\mathfrak{b}^{-1} = \mathbf{Z}w + \mathbf{Z} \cdot 1 = \frac{1}{\lambda} M = \frac{1}{\lambda} \mathfrak{b}_0^{-1} \tag{6}$$

where we defined the ideal $\mathfrak{b}_0 \in \mathfrak{B}$ by $\mathfrak{b}_0^{-1} = \mathrm{M}$. We have

$$\widehat{SL}_2(\mathfrak{o}_K, \mathfrak{b}) = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix} \widehat{SL}_2(\mathfrak{o}_K, \mathfrak{b}_0) \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \tag{7}$$

Instead of looking at the diagonal and at the action of $\widehat{\mathbf{SL}}_2(\mathfrak{o}_K, \mathfrak{b})$ on $\mathfrak{H}^2$, we can consider the action of $\widehat{\mathbf{SL}}_2(\mathfrak{o}_K, \mathfrak{b}_0)$ on $\mathfrak{H}^2$ and the curve $z_1 = \lambda\zeta$, $z_2 = \lambda'\zeta$ in $\mathfrak{H}^2$, where $\zeta \in \mathfrak{H}$. Any totally positive number $\lambda \in M$ can be written uniquely as a linear combination of two consecutive numbers A_{k-1}, A_k with non-negative integers p and q as coefficients (see 2.3, Remark):

$$\lambda = p \cdot A_{k-1} + q \cdot A_k. \tag{8}$$

If we multiply λ by a totally positive unit, then p, q do not change and k only changes modulo r. See the lemma in 2.5 and 2.3 (12). The equation 2.3 (11) shows that the curve $C(\mathfrak{b})$ has in the k-th coordinate system (u_k, v_k) of the resolved cusp the equation

$$(9) \qquad u_k = t^p, v_k = t^q,$$

where t can be restricted to some neighborhood of 0. Namely, we just want to study locally the intersection of our curve with the curves of the resolution. Observe, that p, q are relatively prime because λ is an element of a $\mathbf{Z}$-base of M. The fractional ideals $\mathfrak{b}^{-1} \in \mathfrak{B}^{-1}$ which satisfy our conditions ($\mathfrak{b} \subset \mathfrak{o}_K$ and $\mathfrak{b}$ not divisible by any natural number > 1) are in one-to-one correspondence with the triples $(k \mid p, q)$ where $k \in \mathbf{Z}/r\mathbf{Z}$ and p, q are relatively prime natural numbers and where $(k \mid 0, 1)$ is to be identified with $(k+1 \mid 1, 0)$.

We call $(k \mid p, q)$ *the characteristic of the ideal* $\mathfrak{b} \in \mathfrak{B}$. Actually, k does not stand for an element $k \in \mathbf{Z}/r\mathbf{Z}$, but rather for the corresponding quadratic irrationality w_k which has an invariant meaning. If as in (6)

$$(10) \qquad \mathfrak{b}^{-1} = \mathbf{Z}w + \mathbf{Z} \cdot 1,$$

then (see (8))

$$(11) \qquad w = \frac{\bar{p}\, A_{k-1} + \bar{q}\, A_k}{p\, A_{k-1} + q\, A_k} = \frac{\bar{p}\, w_k + \bar{q}}{p\, w_k + q}$$

where $\begin{pmatrix} \bar{p} & \bar{q} \\ p & q \end{pmatrix} \in \mathbf{SL}_2(\mathbf{Z})$ and $p \geqq 0$, $q \geqq 0$. Therefore, we can determine the characteristic of $\mathfrak{b}$ by writing w in the form (11).

In view of (7) the algebraic surface $\hat{Y}(\mathfrak{o}_K, \mathfrak{b})$ depends only on the ideal class $\mathfrak{B}$. The identification of $\hat{Y}(\mathfrak{o}_K, \mathfrak{b})$ and $\hat{Y}(\mathfrak{o}_K, \mathfrak{b}_0)$ is uniquely defined by (7). We shall denote the surface by $\hat{Y}(\mathfrak{o}_K, \mathfrak{B})$. In a similar way the algebraic surface $Y(\mathfrak{o}_K, \mathfrak{B})$ is defined. The preceding discussions (see in particular (9)) yields the following theorem.

THEOREM. *Let K be a real quadratic field and $\mathfrak{B}$ a narrow ideal class of $\mathfrak{o}_K$. For every ideal $\mathfrak{b} \subset \mathfrak{o}_K$ with $\mathfrak{b} \in \mathfrak{B}$ such that $\mathfrak{b}$ is not divisible by any natural number > 1, we have defined an irreducible curve $C(\mathfrak{b}) = \hat{\pi}(\overline{\mathfrak{H}/\hat{\Gamma}_{\mathfrak{b}}})$ in the non-singular algebraic surface $\hat{Y}(\mathfrak{o}_K, \mathfrak{B})$. The cusp at ∞ of $\overline{\mathfrak{H}/\hat{\Gamma}_{\mathfrak{b}}}$ is mapped by $\hat{\pi}$ to a point on the union of the curves $S_0, \ldots, S_{r-1}$ in $\hat{Y}(\mathfrak{o}_K, \mathfrak{B})$ which were obtained by the resolution of the cusp at ∞ of $\overline{\mathfrak{H}^2/\hat{G}(\mathfrak{o}_K, \mathfrak{b})}$.*

If $\mathfrak{b}$ *has the characteristic* $(k \mid 0, 1)$, *then* $\mathfrak{b}^{-1} = \mathbf{Z}w_k + \mathbf{Z} \cdot 1$ *where* w_k *is the reduced quadratic irrationality belonging to* k, *and the curve* $C(\mathfrak{b})$ *intersects* S_k *transversally in* P *which is not a double point of* $\cup S_j$. *The curve* S_k *is given in the local coordinate system* (u_k, v_k) *by* $v_k = 0$ *and* $C(\mathfrak{b})$ *by* $u_k = 1$. *If* $\mathfrak{b}$ *has the characteristic* $(k \mid p, q)$ *where* $p > 0$ *and* $q > 0$, *then* P *is given in the k-th coordinate system by* $u_k = v_k = 0$, *the curve* S_k *by* $v_k = 0$, *the curve* S_{k-1} *by* $u_k = 0$, *and* $C(\mathfrak{b})$ *has the local equation* $u_k^q = v_k^p$.

If K *has a unit of negative norm, then* $Y(\mathfrak{o}_K, \mathfrak{B}) = \hat{Y}(\mathfrak{o}_K, \mathfrak{B})$. *If* K *does not have such a unit, then in the non-singular algebraic surface* $Y(\mathfrak{o}_K, \mathfrak{B})$ *we have a curve* $D(\mathfrak{b})$ *which in the neighborhood of the resolved cusp at* ∞ *is just the inverse image of* $C(\mathfrak{b})$, *the resolution of the cusp at* ∞ *being an unbranched double cover of the cycle of curves* $S_0, \ldots, S_{r-1}$. *(The fundamental group of a neighborhood of* $S_0 \cup \ldots \cup S_{r-1}$ *is infinite cyclic and we have to take the corresponding covering of degree 2.) The curve* $D(\mathfrak{b})$ *is irreducible or the union of two irreducible curves* $D_1(\mathfrak{b})$, $D_2(\mathfrak{b})$.

Remark. For different $\mathfrak{b}, \hat{\mathfrak{b}} \in \mathfrak{B}$ the curves $C(\mathfrak{b})$, $C(\hat{\mathfrak{b}})$ may coincide. The curve $C(\mathfrak{b}) = \hat{\pi}(\overline{\mathfrak{H}/\hat{\Gamma}_{\mathfrak{b}}})$ may intersect $\cup S_j$ in other points than P which correspond to other cusps of $\overline{\mathfrak{H}/\hat{\Gamma}_{\mathfrak{b}}}$.

4.2. In view of the preceding proposition and the theorem it is important to have a simple method to calculate $N(\mathfrak{b})$ if $\mathfrak{b}^{-1} = \mathbf{Z}w + \mathbf{Z} \cdot 1$. Let D be the discriminant of K (see 1.4), then w can be written uniquely in the form

$$w = \frac{M + \sqrt{D}}{2N} \quad \text{(see 2.6)} \tag{12}$$

where $N > 0$ and $M^2 - D \equiv 0 \bmod 4N$. Then we have

$$N(\mathfrak{b}) = N \tag{13}$$

To prove (13), one checks

$$N \cdot \mathfrak{b}^{-1} \cdot (\mathfrak{b}^{-1})' = (1)$$

If we start with a reduced quadratic irrationality w_0 of the form (12), then the formula

$$w_k = b_k - \frac{1}{w_{k+1}}$$

where $b_k \in \mathbf{Z}$ and $w_{k+1} > 1$, determines inductively for $k \geqq 0$ the b_k and the w_k. We put

(14) $$w_k = \frac{M_k + \sqrt{D}}{2N_k}$$

This is the process of calculating the continued fraction for w_0. If $\mathfrak{b}$ is the ideal of characteristic $(k \mid p, q)$, see (11), then

(15) $$N(\mathfrak{b}) = p^2 N_{k-1} + pqM_k + q^2 N_k, \quad \text{where} \quad M_k^2 - 4N_{k-1}N_k = D,$$

as follows from (11), (13) and (14).

We shall tabulate the values of b_k, M_k, N_k for some w_0, namely for those quadratic irrationalities which are needed later to show that the Hilbert modular surfaces $\overline{\mathfrak{H}^2/G}$ are rational for $d = 2, 3, 5, 6, 7, 13, 15, 17, 21, 33$ (compare the table in 3.9). We also include $w_0 = \dfrac{3+\sqrt{3}}{3}$ which is needed for $\overline{(\mathfrak{H} \times \mathfrak{H}^-)/G}$ in the case $d = 3$ (see 3.12).

If r is the length of the cycle of the quadratic irrationality, we tabulate b_k, M_k, N_k only for $0 \leqq k \leqq r-1$, because they are periodic with period r.

$w_0 = 2 + \sqrt{2}$

$D = 8$

k	0	1
b_k	4	2
M_k	4	4
N_k	1	2

$w_0 = 3 + \sqrt{7}$

$D = 28$

k	0	1
b_k	6	3
M_k	6	6
N_k	1	2

$w_0 = \dfrac{5 + \sqrt{21}}{2}$

$D = 21$

k	0
b_k	5
M_k	5
N_k	1

$w_0 = 2 + \sqrt{3}$

$D = 12$

k	0
b_k	4
M_k	4
N_k	1

$w_0 = \dfrac{5 + \sqrt{13}}{2}$

$D = 13$

k	0	1	2
b_k	5	2	2
M_k	5	5	7
N_k	1	3	3

$w_0 = \dfrac{7 + \sqrt{33}}{2}$

$D = 33$

k	0	1	2	3
b_k	7	2	3	2
M_k	7	7	9	9
N_k	1	4	3	4

$w_0 = \frac{3+\sqrt{5}}{2}$

$D = 5$

k	0
b_k	3
M_k	3
N_k	1

$w_0 = 4+\sqrt{15}$

$D = 60$

k	0
b_k	8
M_k	8
N_k	1

$w_0 = \frac{3+\sqrt{3}}{3}$

$D = 12$

k	0	1
b_k	2	3
M_k	6	6
N_k	3	2

$w_0 = 3+\sqrt{6}$

$D = 24$

k	0	1
b_k	6	2
M_k	6	6
N_k	1	3

$w_0 = \frac{5+\sqrt{17}}{2}$

$D = 17$

k	0	1	2	3	4
b_k	5	3	2	2	3
M_k	5	5	7	9	7
N_k	1	2	4	4	2

4.3. We consider the situation of the theorem in 4.1. Let F be one of the irreducible curves $C(\mathfrak{b})$, $D(\mathfrak{b})$, $D_1(\mathfrak{b})$ or $D_2(\mathfrak{b})$. The curve F has $\overline{\mathfrak{H}/\Gamma}$ as non-singular model where Γ acts effectively on $\mathfrak{H}$ and equals $\hat{\Gamma}_{\mathfrak{b}}/\{1,-1\}$ or $\Gamma_{\mathfrak{b}}/\{1,-1\}$. The curve F lies in a non-singular algebraic surface Y, namely $\hat{Y}(\mathfrak{o}_K, \mathfrak{B})$ or $Y(\mathfrak{o}_K, \mathfrak{B})$. We shall calculate the value of the first Chern class c_1 of Y on F which is up to sign the intersection number of a canonical divisor K of Y with F:

$$c_1[F] = -K\cdot F. \tag{16}$$

The surface Y is a disjoint union of a complex surface (4-dimensional manifold) X with boundary as in 3.4 (19) and open neighborhoods N_ν $(1\leqq \nu \leqq s+t)$ of the configurations of curves into which the s quotient singularities and the t cusp singularities were blown up (minimal resolutions). The first Chern class of X can be represented as in 3.4 by a differential form $\tilde{\gamma}_1$ with compact support in the interior of X and it follows as in 3.4 (25), (26) that

$$\int_F \tilde{\gamma} = \int_F (\omega_1 + \omega_2) \tag{17}$$

where $\omega_j = -\frac{1}{2\pi} y_j^{-2}\, dx_j \wedge dy_j$. Since F comes from the diagonal

$z_1 = z_2$ of $\mathfrak{H}^2$, we obtain that $\int_F \tilde{\gamma}_1$ equals twice the Euler volume of $\mathfrak{H}/\Gamma$. Thus by 1.4 (10)

$$\int_F \tilde{\gamma}_1 = 2 \int_{\mathfrak{H}/\Gamma} \omega = -\frac{1}{3}[G:\Gamma], \tag{18}$$

where $G = \mathbf{SL}_2(\mathbf{Z})/\{1, -1\}$.

We have denoted the open neighborhoods of the resolved quotient singularities and cusps singularities by N_ν $(1 \leqq \nu \leqq s+t)$ where s is the number of quotient singularities and t the number of cusp singularities in the surface $\overline{\mathfrak{H}^2/\hat{G}(\mathfrak{o}_K, \mathfrak{B})}$ or $\overline{\mathfrak{H}^2/G(\mathfrak{o}_K, \mathfrak{B})}$ which has Y as non-singular model. The first Chern class of N_ν can be represented by a differential form $\tilde{\gamma}_1{}^{(\nu)}$ with compact support in N_ν in such a way that $\tilde{\gamma}_1 + \sum \tilde{\gamma}_1{}^{(\nu)}$ represents the first Chern class of Y. By Poincaré duality in N_ν each $\tilde{\gamma}_1{}^{(\nu)}$ corresponds to a linear combination with rational coefficients of the curves into which the singularity was blown up. This linear combination will be called the Chern divisor of the singularity and denoted by $c_1{}^{(\nu)}$. It follows that

$$c_1[F] = 2 \int_{\mathfrak{H}/\Gamma} \omega + \sum_{\nu=1}^{s+t} c_1^{(\nu)} \cdot F \tag{19}$$

We denote the curves of the minimal resolution of a singularity by S_j.

For a quotient singularity the Chern divisor equals $\sum a_j S_j$ where the rational numbers a_j are determined by the linear equations

$$\sum_j (S_i \cdot S_j)\, a_j = 2 + S_i \cdot S_i \tag{20}$$

This follows by the adjunction formula, since all the S_i are rational and non-singular. In some cases we have calculated the numbers a_j at the end of 3.3. For any quotient singularity of type $(p; 1, q)$ the matrix $(S_i \cdot S_j)$ equals

$$\begin{bmatrix} -b_1 & 1 & 0 & \cdots & & 0 \\ 1 & -b_2 & 1 & 0 & \cdots & \vdots \\ 0 & 1 & \ddots & & & \vdots \\ \vdots & & & \ddots & & 1 \\ 0 & \cdots & & & 1 & -b_r \end{bmatrix} \qquad \begin{array}{l} b_i \geqq 2 \\ b_i = -S_i \cdot S_i \end{array}$$

where $p/q = [[b_1, ..., b_r]]$, (see [35], 3.4).

The inverse of this matrix has only non-positive entries. Since $2 + S_i \cdot S_i = 2 - b_i \leqq 0$, we have $a_j \geqq 0$.

For a cusp singularity the Chern divisor equals $\sum S_j$, (see 3.2 (9)). Therefore, in (19) all the terms $c_1^{(\nu)} \cdot F$ are non-negative.

Every cusp of $\overline{\mathfrak{H}/\Gamma}$, the non-singular model of F, maps under $\overline{\mathfrak{H}/\Gamma} \to Y$ to a point on some curve in the Chern divisor of a cusp singularity. This intersection point gives at least the contribution 1 in (19).

Let $a_r(\Gamma)$ be defined as in 1.6. If an element γ of Γ has order r, then (since $\Gamma \subset \hat{G}(\mathfrak{o}_K, \mathfrak{b})$ or $\Gamma \subset G(\mathfrak{o}_K, \mathfrak{b})$) we have a quotient singularity of type $(r; 1, 1)$ whose Chern divisor intersects F in a point coming by $\mathfrak{H} \to \mathfrak{H}/\Gamma \to F$ from a point z of $\mathfrak{H}$ whose isotropy group is generated by γ. The Chern divisor contains in this case just one curve S and equals $\frac{r-2}{r} S$, (see the end of 3.3).

If we denote by $\sigma(\Gamma)$ the number of cusps of $\overline{\mathfrak{H}/\Gamma}$ we get by (19) the estimate

$$c_1[F] \geqq 2\int_{\mathfrak{H}/\Gamma} \omega + \sum_{r \geqq 2} \frac{r-2}{r} a_r(\Gamma) + \sigma(\Gamma) \tag{20}$$

The Euler number of the non-singular model $\overline{\mathfrak{H}/\Gamma}$ of F is given by the classical formula

$$e(\overline{\mathfrak{H}/\Gamma}) = \int_{\mathfrak{H}/\Gamma} \omega + \sum_{r \geqq 2} \frac{r-1}{r} a_r(\Gamma) + \sigma(\Gamma), \tag{21}$$

which follows from 1.6 (21), because $\sigma(\Gamma)$ points are attached to $\mathfrak{H}/\Gamma$ by the compactification. By (20) and (21)

$$c_1[F] \geqq 2e(\overline{\mathfrak{H}/\Gamma}) - \sum_{r \geqq 2} a_r(\Gamma) - \sigma(\Gamma) \tag{22}$$

The right side of (20) is defined for any discrete subgroup of type (F) which is equivalent in this case to $\mathfrak{H}/\Gamma$ having a finite volume.

Definition:

$$c_1(\Gamma) = 2\int_{\mathfrak{H}/\Gamma} \omega + \sum_{r \geqq 2} \frac{r-2}{r} a_r(\Gamma) + \sigma(\Gamma)$$
$$= 2e(\overline{\mathfrak{H}/\Gamma}) - \sum_{r \geqq 2} a_r(\Gamma) - \sigma(\Gamma).$$

If Γ is the Klein-Fricke group $\Gamma_0(N)$ divided by $\{1, -1\}$ we shall write $c_1(N)$ for $c_1(\Gamma)$ and also $a_r(N)$ for $a_r(\Gamma)$ and $\sigma_0(N)$ for $\sigma(\Gamma)$. The numbers $a_r(N)$ vanish for $r > 3$. There are well-known formulas for $[\mathbf{SL}_2(\mathbf{Z}) : \Gamma_0(N)]$, for $a_r(N)$ and $\sigma_0(N)$, (see, for example, [70], p. 24). The Euler number $e(\mathfrak{H}/\Gamma_0(N))$ will be written as $2 - 2g_0(N)$. By (21) there is a formula for $g_0(N)$ which implies (as Helling has shown recently [32])

$$(23) \qquad \begin{aligned} g_0(N) &= 0 \Leftrightarrow N = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 16, 18, 25 \\ g_0(N) &= 1 \Leftrightarrow N = 11, 14, 15, 17, 19, 20, 21, 24, 27, 32, 36, 49 \end{aligned}$$

Compare [13] where the values of $g_0(N)$, $a_r(N)$ and $\sigma_0(N)$ are tabulated for $N \leqq 1\,000$. Therefore, we can write down easily a list of $c_1(N)$ for the *rational and elliptic* curves $\mathfrak{H}/\Gamma_0(N)$ (see (23)):

(24)

$g_0(N) = 0$

N	1	2	3	4	5	6	7	8	9	10	12	13	16	18	25
$c_1(N)$	1	1	1	1	0	0	0	0	0	−2	−2	−2	−2	−4	−4

$g_0(N) = 1$

N	11	14	15	17	19	20	21	24	27	32	36	49
$c_1(N)$	−2	−4	−4	−4	−4	−6	−6	−8	−6	−8	−12	−10

4.4. We want to prove that the Hilbert modular surfaces are rational in some cases. An algebraic surface is rational if and only if it is birationally equivalent to the complex projective plane, or equivalently if the field of meromorphic functions on the surface is a purely transcendental extension of the field of complex numbers of degree 2.

Let S be a non-singular algebraic surface and K a canonical divisor of S. The "complete linear system" $|mK|$ of all non-negative divisors D which are linearly equivalent to mK is a complex projective space whose dimension is denoted by $P_m - 1$. The numbers P_m $(m \geqq 1)$ are the plurigenera of the surface S (see, for example, [64] and [36], p. 151).

We have $P_1 = g_2$ (see 3.6). The equality $P_m = 0$ means, that $|mK|$ is empty. The numbers P_m $(m \geqq 1)$ are birational invariants. They vanish for rational surfaces.

Castelnuovo's criterion ([46], Part IV):

A non-singular connected algebraic surface S is rational if and only if $g_1 = P_2 = 0$.

Remark. Clearly, $P_2 = 0$ implies $g_2 = 0$. There are algebraic surfaces with $g_1 = g_2 = 0$ which are not rational (Enriques' surfaces with $g_1 = g_2 = 0$ and $P_2 = 1$, see [64]). The condition $g_1 = g_2 = 0$ is equivalent to $g_1 = 0$ and $\chi(S) = 1$ (see 3.6). For Hilbert modular surfaces $g_1 = 0$ (see the lemma in 3.6). Up to now all Hilbert modular surfaces and similar surfaces (see § 5) with $\chi(S) = 1$ have turned out to be rational. The number P_m of a non-singular model of $\overline{\mathfrak{H}^2/\Gamma}$ equals the dimension of the vector space of those cusp forms of weight m which can be extended holomorphically to the non-singular model. Therefore $P_m \leqq \dim \mathfrak{S}_\Gamma(m)$. The calculation of P_m seems to be a very difficult problem.

We shall base everything on Castelnuovo's criterion, not worrying whether in a systematic exposition of the theory of algebraic surfaces some results would have to be presented before this criterion. The following theorem is an immediate consequence of Castelnuovo's criterion.

THEOREM. *Let S be a non-singular connected algebraic surface with* $g_1 = 0$. *Let* c_1 *be the first Chern class of S and K a canonical divisor of S. If D is an irreducible curve in S with* $c_1[D] = -K \cdot D > 0$ *and* $D \cdot D \geqq 0$, *then S is rational.*

Proof. We show that $P_m = 0$ for $m \geqq 1$.

If $A \in |mK|$, then

$$A = aD + R, \quad \text{where} \quad a \geqq 0,\ R \cdot D \geqq 0.$$

Therefore,

$$mK \cdot D = aD \cdot D + R \cdot D \geqq 0$$

which is a contradiction. Thus mK is empty.

COROLLARY I. *Let S be a non-singular connected algebraic surface with* $g_1 = 0$. *Let* c_1 *be the first Chern class of S and K a canonical divisor of S. If D is an irreducible curve on S with* $c_1[D] \geqq 2$, *then S is rational. If D is an irreducible curve on S with* $c_1[D] \geqq 1$ *which has at least one singular point or which is not a rational curve, then S is rational.*

Proof. By the adjunction formula (0.6)

$$c_1[D] - D \cdot D = - K \cdot D - D \cdot D$$

equals the Euler number $e(\tilde{D})$ of the non-singular model $\tilde{D}$ of D minus contributions coming from the singular points of D which are positive and even for each singular point. Thus

$$c_1[D] - D \cdot D \leqq e(\tilde{D}) \leqq 2$$

and

$$c_1[D] - D \cdot D \leqq 0,$$

if D has a singular point or is not rational. Therefore, the assumptions in the corollary imply $D \cdot D \geqq 0$.

COROLLARY II. *Let S be a non-singular connected algebraic surface with $g_1 = 0$. Let c_1 be the first Chern class of S. Suppose that S is not a rational surface. If D is an irreducible curve on S with $c_1[D] = - K \cdot D = 1$, then D is rational and does not have a singular point. Furthermore, $D \cdot D = - 1$.*

A non-singular rational curve E on a non-singular surface S which satisfies $E \cdot E = - 1$ (or equivalently $c_1[E] = 1$) is called an exceptional curve (of the first kind). It can be blown down to a point:

In a natural way, S/E is again an algebraic surface ([64], p. 32). The surfaces S and S/E are birationally equivalent.

If c_1 is the first Chern class of S and $\tilde{c}_1$ the first Chern class of S/E, then for any irreducible curve D in S and the image curve $\tilde{D}$ in S/E we have

$$\tilde{c}_1[\tilde{D}] = c_1[D] + D \cdot E \tag{25 a}$$

This is true because $c_1 = \pi^*\tilde{c}_1 - e$, where $\pi : S \to S/E$ is the natural map and $e \in H^2(S, \mathbf{Z})$ the cohomology class corresponding to E under Poincaré duality.

If D is non-singular and $D \cdot E = 1$, then $\tilde{D}$ is also non-singular and by (25 a) and the adjunction formula

$$D \cdot D = \tilde{D} \cdot \tilde{D} - 1 \tag{25 b}$$

COROLLARY III. *Let S be a non-singular algebraic surface with $g_1 = 0$ which is not rational. If E_1, E_2 are two different exceptional curves of the first kind, then E_1, E_2 do not intersect.*

Proof. We have $c_1 [E_1] = 1$. If we blow down E_2, then in S/E_2 (first Chern class $\tilde{c}_1$)

$$\tilde{c}_1(\tilde{E}_1) = 1 + E_1 \cdot E_2$$

Therefore, by Corollary I, $E_1 \cdot E_2 = 0$ and thus $E_1 \cap E_2 = \emptyset$.

4.5. Let G be the Hilbert modular group for $K = \mathbf{Q}(\sqrt{d})$, d square free. If we resolve all singularities in $\overline{\mathfrak{H}^2/G}$ (minimal resolutions) we get a non-singular algebraic surface $Y(d)$ which in 4.1 was denoted by $Y(\mathfrak{o}_K, \mathfrak{B})$ where $\mathfrak{B}$ is here the ideal class of principal ideals$(\lambda) \subset \mathfrak{o}_K$ with $\lambda > 0$, $\lambda' > 0$. If λ is not divisible by a natural number > 1, we can consider the curve

$$z_1 = \lambda\zeta,\ z_2 = \lambda'\zeta \quad (\zeta \in \mathfrak{H}) \tag{26}$$

which according to (7) gives one of the (one or two) irreducible components of the curve $D((\lambda))$ in $Y(d)$. If we replace λ by λ' we get the same curve. Namely, our curve can also be written as

$$z_1 = \lambda\left(-\frac{1}{\lambda\lambda'\zeta}\right), \quad z_2 = \lambda'\left(-\frac{1}{\lambda\lambda'\zeta}\right), \ \zeta \in \mathfrak{H}, \tag{27}$$

because $\zeta \mapsto -\dfrac{1}{\lambda\lambda'\zeta}$ is an automorphism of $\mathfrak{H}$.

If we apply the element $z_j \mapsto -\dfrac{1}{z_j}$ of G to (27) we get

$$z_1 = \lambda'\zeta, \quad z_2 = \lambda\zeta,\ \zeta \in \mathfrak{H}. \tag{28}$$

We consider the involution $(z_1, z_2) \mapsto (z_2, z_1)$ on $\mathfrak{H}^2$ which induces an involution T on $\overline{\mathfrak{H}^2/G}$ and hence on $Y(d)$, because the minimal resolutions are canonical. (26) and (28) show that our curve is carried over to itself by T.

The cusp at ∞ of $\overline{\mathfrak{H}^2/G}$ admits a resolution:

We have to take the primitive cycle or twice the primitive cycle of the reduced quadratic irrationality w_0 such that $\mathbf{Z}w_0 + \mathbf{Z} \cdot 1 = \mathfrak{o}_K$:

$$(29) \qquad w_0 = [\sqrt{d}] + 1 + \sqrt{d} \qquad \text{for } d \equiv 2, 3 \bmod 4$$
$$w_0 = \tfrac{1}{2}(\{\sqrt{d}\} + \sqrt{d}) \qquad \text{for } d \equiv 1 \bmod 4$$

where $\{\sqrt{d}\}$ denotes the smallest odd number greater than $\sqrt{d}$. We have $(w_1^{-1})' = w_0$, (see 4.2) and therefore (2.3 (13)) for the continued fraction of w_0:

$$(30) \quad w_{-k} = w_k,\ b_k = b_{-k},\ M_k = M_{-k},\ N_k = N_{-k},\ (w_k^{-1})' = w_{-k+1}$$

If r is the length of the cycle, then for w_k, b_k, M_k, N_k the index k can be taken mod r. However, for the curves S_k we have to consider k modulo r or modulo $2r$.

We note

$$(31) \quad b_0 = M_0 = 2([\sqrt{d}] + 1),\ N_0 = 1,\ N_1 = ([\sqrt{d}] + 1)^2 - d \qquad \text{for } d \equiv 2, 3 \bmod 4$$

$$(32) \quad b_0 = M_0 = \{\sqrt{D}\},\ N_0 = 1,\ N_1 = \frac{1}{4}(\{\sqrt{d}\}^2 - d) \qquad \text{for } d \equiv 1 \bmod 4$$

For any characteristic $(k \mid p, q)$ we have one or two curves (26) in the Hilbert modular surface $Y(d)$. Compare the theorem in 4.1. Let D be such a curve. Suppose

$$(33) \qquad N = N(\lambda) = p^2 N_{k-1} + pq\, M_k + q^2 N_k \not\equiv 0 \mod d,$$

then the non-singular model of D is $\mathfrak{H}/\Gamma_0(N)$. Suppose also $N > 1$. Then the curve D intersects the Chern divisor $\cup S_j$ of the resolution at least twice, the intersection points correspond to the cusp at ∞ and at 0 of $\mathfrak{H}/\Gamma_0(N)$ which are different cusps for $N > 1$. By applying the theorem in 4.1 to the curves (26) and (28) which both represent D we see by (11) that the two intersection points are of characteristic $(k \mid p, q)$ and $(-k+1 \mid q, p)$. The involution T maps S_k to S_{-k} and interchanges the two intersection points. If the characteristic is $(k \mid 0, 1)$, then the symmetric one is $(-k+1 \mid 1, 0) = (-k \mid 0, 1)$. If (33) is satisfied, then

$$c_1[D] \geqq c_1(N), \quad \text{see (24)},$$

because the non-singular model of D is $\overline{\mathfrak{H}/\Gamma_0(N)}$.

Since the intersection number of $u^q - v^p = 0$ and $uv = 0$ equals $p + q$, the intersection number of D and the Chern divisor $\sum S_j$ is $\geqq p + q$ in each of the two intersection points and therefore

$$c_1[D] \geqq c_1(N) + 2(p+q-1) \tag{34}$$

Because of 4.4 (Corollary I) we have the following theorem.

THEOREM. *Let $K = \mathbf{Q}(\sqrt{d})$, d square free, and G the Hilbert modular group. Consider the continued fraction for w_0 (see* (29)*) and the corresponding numbers M_k, N_k (see* 4.2*). We look at the following representations of natural numbers N:*

$$N = p^2 N_{k-1} + pqM_k + q^2 N_k \tag{35}$$

(for some k and for relative prime natural numbers p, q).

If N is represented as in (35), *if $N \not\equiv 0 \bmod d$ and $N > 1$, then*

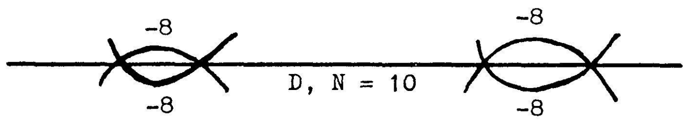
$$c_1(N) + 2(p+q-1) < 2$$

or the Hilbert modular suface $\overline{\mathfrak{H}^2/G}$ is rational.

$N_{-1} + M_0 + N_0$ equals 7 for $d = 2,21$, it equals 8 for $d = 17$, it equals 9 for $d = 7,13$ (see 4.2 or recall that $N_{-1} = N_1$ and use (31), (32)). For $d = 3$, we have $13 = 4N_1 + 2M_0 + N_0$. For $d = 5$, we have $11 = 4N_1 + 2M_0 + N_0$. For these d we get $c_1(N) + 2(p+q-1) = 2$ (see (24)). Thus the Hilbert modular surface is rational in these cases.

For $d = 6, 15, 33$ a more refined argument is needed. Actually, the theorem throws away some information, because we have only used two cusps of $\overline{\mathfrak{H}/\Gamma_0(N)}$, $(N>1)$. If N is not a prime, then $\overline{\mathfrak{H}/\Gamma_0(N)}$ has more cusps. This is relevant for $d = 15$: There are two cusps of the Hilbert modular surface which are of equal type (3.9). We have $10 = N_{-1} + M_0 + N_0$. The curve $\overline{\mathfrak{H}/\Gamma_0(10)}$ has 4 cusps. One can prove that the intersection of D with the Chern cycles of the two cusps of the Hilbert modular surface looks as follows (in this case the curve $D(\mathfrak{b})$ of theorem 4.1 is irreducible)

(35)

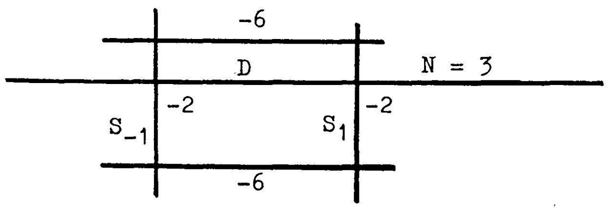


Therefore

$$c_1[D] \geqq c_1(10) + 4 = 2.$$

For $d = 6$ we have a diagram

(37)



Again the curve $D = D(\mathfrak{b})$ of the theorem in 4.1 is irreducible.

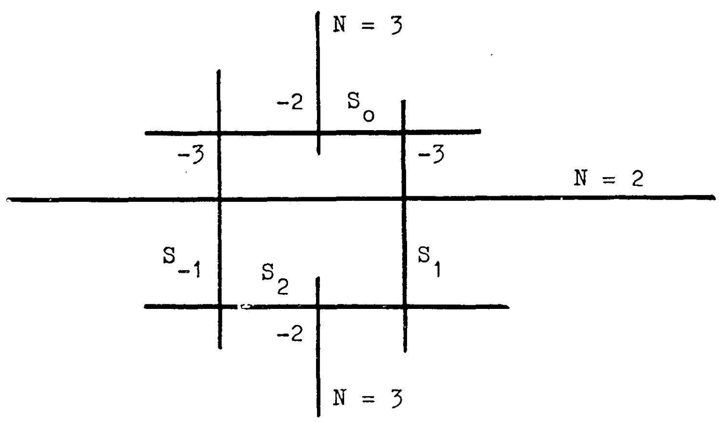
For the curve D we have $c_1[D] \geqq c_1(N) = 1$. Thus the surface $Y(6)$ is rational or D is an exceptional curve of the first kind. If D is exceptional, then we blow it down. The images of S_1 and S_{-1} become exceptional curves which intersect each other. Thus $Y(6)$ is rational by Corollary III in 4.4. We could have also used $N = 10$. The corresponding curve goes through the 4 corners of diagram (37).

For $d = 33$, the same argument works using $N = 4$.

We have proved

THEOREM. *Let $K = \mathbf{Q}(\sqrt{d})$, d square free, and G the Hilbert modular group, then $\overline{\mathfrak{H}^2/G}$ is rational for $d = 2, 3, 5, 6, 7, 13, 15, 17, 21, 33$.*

For $d = 3$ we consider also $\overline{(\mathfrak{H} \times \mathfrak{H}^-)/G}$. The non-singular model is $Y(\mathfrak{o}_K, \mathfrak{B})$ where $\mathfrak{B}$ is now the ideal class of all ideals (λ) with $\lambda\lambda' < 0$. The resolution of the cusp at infinity is



We have one curve with $N = 2$ (non-singular model $\mathfrak{H}/\Gamma_0(2)$) and two curves with $N = 3$ (non-singular model $\mathfrak{H}/\Gamma^*(3)$).

If $\Gamma = \Gamma^*(3)/\{1, -1\}$, then $e(\overline{\mathfrak{H}^2/\Gamma}) = 2$, $a_2(\Gamma) = a_6(\Gamma) = 1$, all other $a_r(\Gamma) = 0$, $\sigma(\Gamma) = 1$. Thus

$$c_1(\Gamma^*(3)) = 4 - 2 - 1 = 1$$

Either the surface is rational, or the three curves with $N = 2, 3$ can be blown down. Then S_0 can be blown down and S_1 and S_{-1} give two exceptional curves which intersect in two points. Thus the surface is rational.

Observe that in general the rationality of $Y(\mathfrak{o}_K, \mathfrak{B})$ implies the rationality of $\hat{Y}(\mathfrak{o}_K, \mathfrak{B})$ (Lüroth's theorem [64], Chap. III, § 2). We could show this directly by using our curves in $\hat{Y}(\mathfrak{o}_K, \mathfrak{B})$.

Exercise. Let $K = \mathbf{Q}(\sqrt{69})$. Calculate the arithmetic genera of $\overline{\mathfrak{H}^2/G}$ and $\overline{\mathfrak{H}^2/\hat{G}}$. Prove that the surface $\overline{\mathfrak{H}^2/\hat{G}}$ is rational !

In all cases where we know that the arithmetic genus equals 1 *we have proved rationality.*

§ 5. The symmetric Hilbert modular group for primes $p \equiv 1 \bmod 4$

5.1. Let S be a compact connected non-singular algebraic surface. The fixed point set D of a holomorphic involution T of S (different from the identity) consist of finitely many isolated fixed points $P_1, \ldots, P_r$ and a disjoint union of connected non-singular curves $D_1, \ldots, D_s$.

If there are no isolated fixed points P_j, then S/T is non-singular and the arithmetic genera of S and S/T are related by the formula

$$\chi(S/T) = \frac{1}{2}\left(\chi(S) + \frac{1}{4} c_1 [D]\right) \tag{1}$$

where $D = \sum D_i$ and c_1 is the first Chern class of S (see [40], § 3).

Furthermore, if F is a curve on S (not necessarily irreducible) with $T(F) = F$ and F not contained in D and if $\tilde{F}$ is the image curve on S/T, then

$$\tilde{c}_1 [\tilde{F}] = \tfrac{1}{2}(c_1 [F] + F \cdot D), \quad \text{where } \tilde{c}_1 = \text{first Chern class of } S/T. \tag{2}$$

Proof. If $\pi : S \to S/T$ is the natural projection, then $c_1 = \pi^* \tilde{c}_1 - d$ where $d \in H^2(S, \mathbf{Z})$ is the Poincaré dual of the branching divisor D. Thus

$$(c_1 + d)[F] = \tilde{c}_1 [2\tilde{F}].$$