

# §5. The symmetric Hilbert modular group for primes $p \equiv 1 \pmod{4}$

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Either the surface is rational, or the three curves with  $N = 2, 3$  can be blown down. Then  $S_0$  can be blown down and  $S_1$  and  $S_{-1}$  give two exceptional curves which intersect in two points. Thus the surface is rational.

Observe that in general the rationality of  $Y(\mathfrak{o}_K, \mathfrak{B})$  implies the rationality of  $\hat{Y}(\mathfrak{o}_K, \mathfrak{B})$  (Lüroth's theorem [64], Chap. III, § 2). We could show this directly by using our curves in  $\hat{Y}(\mathfrak{o}_K, \mathfrak{B})$ .

*Exercise.* Let  $K = \mathbf{Q}(\sqrt{69})$ . Calculate the arithmetic genera of  $\overline{\mathfrak{S}^2/G}$  and  $\overline{\mathfrak{S}^2/\hat{G}}$ . Prove that the surface  $\overline{\mathfrak{S}^2/\hat{G}}$  is rational !

*In all cases where we know that the arithmetic genus equals 1 we have proved rationality.*

### § 5. THE SYMMETRIC HILBERT MODULAR GROUP FOR PRIMES $p \equiv 1 \pmod{4}$

5.1. Let  $S$  be a compact connected non-singular algebraic surface. The fixed point set  $D$  of a holomorphic involution  $T$  of  $S$  (different from the identity) consist of finitely many isolated fixed points  $P_1, \dots, P_r$  and a disjoint union of connected non-singular curves  $D_1, \dots, D_s$ .

If there are no isolated fixed points  $P_j$ , then  $S/T$  is non-singular and the arithmetic genera of  $S$  and  $S/T$  are related by the formula

$$(1) \quad \chi(S/T) = \frac{1}{2} \left( \chi(S) + \frac{1}{4} c_1 [D] \right)$$

where  $D = \sum D_i$  and  $c_1$  is the first Chern class of  $S$  (see [40], § 3).

Furthermore, if  $F$  is a curve on  $S$  (not necessarily irreducible) with  $T(F) = F$  and  $F$  not contained in  $D$  and if  $\tilde{F}$  is the image curve on  $S/T$ , then

$$(2) \quad \tilde{c}_1 [\tilde{F}] = \frac{1}{2} (c_1 [F] + F \cdot D), \quad \text{where } c_1 = \text{first Chern class of } S/T.$$

*Proof.* If  $\pi : S \rightarrow S/T$  is the natural projection, then  $c_1 = \pi^* \tilde{c}_1 - d$  where  $d \in H^2(S, \mathbf{Z})$  is the Poincaré dual of the branching divisor  $D$ . Thus

$$(c_1 + d) [F] = \tilde{c}_1 [2\tilde{F}].$$

5.2. Let  $p$  be a prime  $\equiv 1 \pmod{4}$ . We consider the field  $K = \mathbf{Q}(\sqrt{p})$  and its Hilbert modular group  $G$ . We make these restrictions throughout § 5 though some of our results are valid more generally.

The involution  $(z_1, z_2) \mapsto (z_2, z_1)$  induces an involution  $T$  of  $\mathfrak{H}^2/G$  and of  $\overline{\mathfrak{H}^2/G}$ . As mentioned before (4.5), it can be lifted to an involution  $T$  of our non-singular model  $Y(p)$  because this was obtained by the canonical minimal resolution of all singularities in  $\overline{\mathfrak{H}^2/G}$ .

We shall study the algebraic surface  $Y(p)/T$  (the isolated fixed points of  $T$  give rise to quotient singularities of type  $(2; 1, 1)$  of this surface), calculate its arithmetic genus and determine for which  $p$  the surface is rational (see [39], [40]).

Equivalently we can consider the *symmetric Hilbert modular* group  $G_T$  which is an extension of index 2 of  $G$  by the involution  $(z_1, z_2) \mapsto (z_2, z_1)$  and study  $\mathfrak{H}^2/G_T$ :

The surface  $Y(p)/T$  (with the quotient singularities resolved) is a non-singular model of the compactification of  $\mathfrak{H}^2/G_T$ .

5.3. The field  $K$  has a unit of negative norm. Therefore, the groups  $G$  and  $\hat{G}$  coincide (1.7). The class number of  $K$  is odd. The ideal class groups  $C$  and  $C^+$  are equal and the homomorphism  $Sq$  in 3.7 (42) is an isomorphism. Therefore for any ideal  $\mathfrak{b} \subset \mathfrak{o}_K$  we can find a matrix  $A \in \mathbf{GL}_2^+(K)$  (see 1.3) such that

$$(3) \quad A^{-1} \mathbf{SL}_2(\mathfrak{o}_K) A = \mathbf{SL}_2(\mathfrak{o}_K, \mathfrak{b})$$

(see 3.7 (40) and 4.1 (7)). If  $A_1, A_2$  are matrices satisfying (3), then, for  $B = A_1 A_2^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , we have  $B \mathbf{SL}_2(\mathfrak{o}_K) B^{-1} = \mathbf{SL}_2(\mathfrak{o}_K)$ .

*Proposition.*

If  $B \in \mathbf{GL}_2^+(K)$  and  $B \mathbf{SL}_2(\mathfrak{o}_K) B^{-1} = \mathbf{SL}_2(\mathfrak{o}_K)$ , then

$$(4) \quad \sqrt{\det B} \in K, \frac{1}{\sqrt{\det B}} \cdot B \in \mathbf{SL}_2(\mathfrak{o}_K)$$

*Proof* (compare Maaß [54]). Put  $h^2 = \det B$ . We may assume that  $B$  is an integral matrix. Since

$$\mathbf{SL}_2(\mathfrak{o}_K) \ni B \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} B^{-1} = \begin{pmatrix} 1 - ac/h^2 & a^2/h^2 \\ -c^2/h^2 & 1 + ac/h^2 \end{pmatrix}$$

and a similar formula holds for  $B \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} B^{-1}$ , we see that  $\frac{1}{h} B$  has coefficients which are algebraic integers. Thus the ideal  $(a, c)$  of  $\mathfrak{o}_K$  consists exactly of those elements  $x$  in  $\mathfrak{o}_K$  such that  $x/h$  is an algebraic integer. This implies that  $(a, c)^2$  equals the principal ideal  $(\det B)$ . In our case, the ideal class group has odd order. Thus  $(a, c)$  is principal and  $\det B$  multiplied with a totally positive unit is a square in  $\mathfrak{o}_K$ . But every totally positive unit is a square of a unit. Therefore  $h \in \mathfrak{o}_K$ . For the algebraic number theory needed, see [30], § 37.

An ideal is called *admissible* if it is not divisible by any natural number  $> 1$ . For any admissible ideal  $\mathfrak{b} \subset \mathfrak{o}_K$  we have (4.1) a curve  $C(\mathfrak{b})$  on  $Y(\mathfrak{o}_K, \mathfrak{b}) = \hat{Y}(\mathfrak{o}_K, \mathfrak{b})$ . In view of (3) we have a curve (which we also call  $C(\mathfrak{b})$ ) on our Hilbert modular surface  $Y(p)$ . The curve is given in  $\mathfrak{H}^2/G$  by

$$(5) \quad z_1 = A\zeta, \quad z_2 = A'\zeta, \quad \zeta \in \mathfrak{H}.$$

Because of (4) it does not depend on the choice of  $A$ . (Multiplication of  $A$  from the left by an element of  $\mathbf{SL}_2(\mathfrak{o}_K)$  does not change the curve.)

We can also say that the surfaces  $\mathfrak{H}^2/\mathbf{SL}_2(\mathfrak{o}_K, \mathfrak{b})$  are canonically identified and the curves  $C(\mathfrak{b})$  are the diagonals in the different representations of  $\mathfrak{H}^2/G$  as  $\mathfrak{H}^2/\mathbf{SL}_2(\mathfrak{o}_K, \mathfrak{b})$ . If we change  $A$  by multiplying from the right by a rational matrix with positive determinant, we get the same curve, because we make just a change of the parameter  $\zeta \in \mathfrak{H}$ . This implies that  $C(\mathfrak{b}_1) = C(\mathfrak{b}_2)$  if there exists a matrix  $A_0 \in \mathbf{GL}_2^+(\mathbf{Q})$  such that  $A_0 \mathbf{SL}_2(\mathfrak{o}_K, \mathfrak{b}_1) A_0^{-1} = \mathbf{SL}_2(\mathfrak{o}_K, \mathfrak{b}_2)$ .

*Lemma I.* If  $\mathfrak{b}_1, \mathfrak{b}_2$  are admissible ideals in  $\mathfrak{o}_K$ , then the curves  $C(\mathfrak{b}_1), C(\mathfrak{b}_2)$  coincide if and only if  $N(\mathfrak{b}_1) = N(\mathfrak{b}_2)$ .

*Proof.* If  $N(\mathfrak{b}_1) = N(\mathfrak{b}_2) = N$ , then put  $d = N/N((\mathfrak{b}_1, \mathfrak{b}_2))$ . We have  $(d, N/d) = 1$ , because the ideals are admissible. Thus there exists a rational matrix of determinant  $d$  of the form

$$(6) \quad A_0 = \begin{pmatrix} \alpha_0 d & \beta_1 \\ \gamma_0 & \delta_0 d \end{pmatrix}, \quad \gamma_0 \equiv 0 \pmod{N}$$

where  $\alpha_0, \beta_1, \gamma_0, \delta_0$  are integers. (Such a matrix occurred in a related context in 4.1). Then, for any  $A_0$  with these properties,

$$A_0 \mathbf{SL}_2(\mathfrak{o}_K, \mathfrak{b}_1) A_0^{-1} = \mathbf{SL}_2(\mathfrak{o}_K, \mathfrak{b}_2)$$



which shows that the curves coincide. If the curves coincide, then the norms are equal. (We leave the proof to the reader.)

A natural number  $N \geq 1$  is called admissible (with respect to  $p$ ) if it is the norm of an admissible ideal. The prime ideal theory of quadratic fields which we always have used tacitly yields the following lemma.

*Lemma II.* The natural number  $N \geq 1$  is admissible with respect to  $p$  if and only if  $N$  is not divisible by  $p^2$  and not by any prime  $q \neq p$  with  $\left(\frac{q}{p}\right) = -1$ .

*Definition.*

In view of Lemma I we have a well-defined curve for any admissible natural number  $N$ . This curve on the surface  $Y(p)$  will be called  $F_N$ .

*Lemma III.* For the involution  $T$  of  $Y(p)$  and any admissible  $N$  we have  $T(F_N) = F_N$ .

*Proof.* If  $N = N(\mathfrak{b})$ , then  $F_N = C(\mathfrak{b})$  is given in  $\mathfrak{S}^2/G$  by (5) where  $A$  is as in (3). Therefore  $T(F_N)$  is the curve  $z_1 = A'\zeta$ ,  $z_2 = A\zeta$ . But this is  $C(\mathfrak{b}')$  which equals  $C(\mathfrak{b})$  by lemma I.

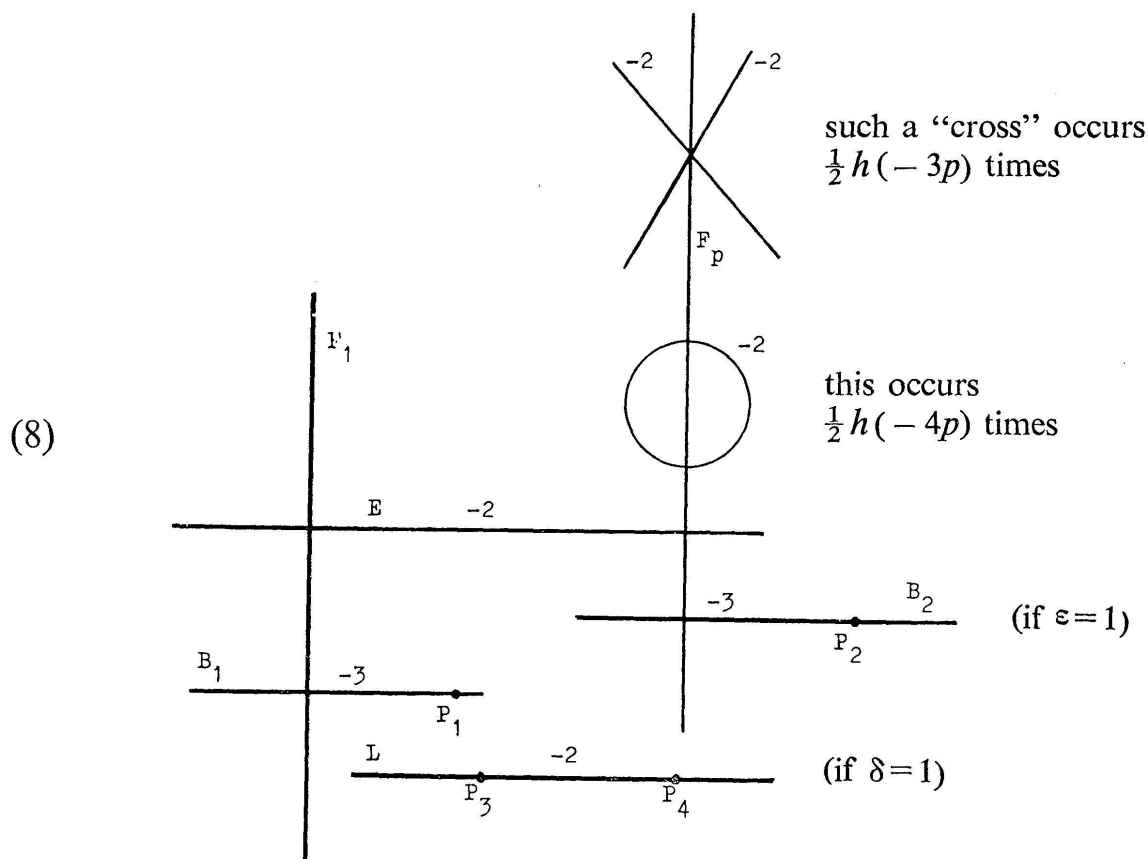
*Remark.* If  $N \not\equiv 0 \pmod p$ , then  $N((\mathfrak{b}, \mathfrak{b}')) = 1$  and the involution  $T$  on  $F_N$  can be given by the matrix  $A_0 = \begin{pmatrix} 0 & 1 \\ N & 0 \end{pmatrix}$  (see (6)) if we lift  $T$  to the non-singular model  $\overline{\mathfrak{S}/\Gamma_0(N)}$  of  $F_N$ . Thus  $\overline{\mathfrak{S}/\Gamma_*(N)}$  is the non-singular model of  $F_N/T$ . (see 4.1). In particular,  $T$  is not the identity on  $F_N$  if  $N \not\equiv 0 \pmod p$  and  $N > 1$ .

5.4. The curves  $F_1$  and  $F_p$  (considered as curves in  $\mathfrak{S}^2/G$ ) are the only curves which are fixed pointwise under  $T$ , (see [14] Part II, [62]). The curve  $F_p$  belongs to the ideal  $(\sqrt{p} e_0)$  where  $e_0$  is a unit of negative norm and can be given by  $z_1 = \sqrt{p} e_0 \zeta$ ,  $z_2 = -\sqrt{p} e_0' \zeta$  or by  $z_1 = e_0^2 z_2$ .

The involution  $T$  acts on the quotient singularities of  $\mathfrak{S}^2/G$ . The description of this action [62] depends on the residue class of  $p \pmod{24}$ . Therefore we define

$$(7) \quad \begin{aligned} \varepsilon &= 1 \text{ for } p \equiv 1 \pmod 3, \quad \varepsilon = 0 \text{ for } p \equiv 2 \pmod 3 \\ \delta &= 1 \text{ for } p \equiv 1 \pmod 8, \quad \delta = 0 \text{ for } p \equiv 5 \pmod 8 \end{aligned}$$

In  $\mathfrak{S}^2/G$  the following holds [62]: Of the  $h(-4p)$  quotient singularities of order 2, half of them lie on  $F_p$  and not on  $F_1$ , and one of them lies on  $F_1$  and  $F_p$  and is the only intersection point of  $F_1$  and  $F_p$  in  $\mathfrak{S}^2/G$ . There are in addition  $\delta$  quotient singularities of order 2 which are fixed under  $T$ . “They” lie neither on  $F_1$  nor on  $F_p$ . The remaining order 2 singularities are interchanged pairwise under  $T$ . Of the  $h(-3p)$  quotient singularities of order 3, exactly half of them are of type  $(3; 1, 2)$ . They lie on  $F_p$ . There is one singularity of type  $(3; 1, 1)$  which lies on  $F_1$  whereas  $\varepsilon$  such singularities lie on  $F_p$ . The remaining singularities of type  $(3; 1, 1)$  are interchanged pairwise. For  $p = 5$ , the two singularities of order 5 are interchanged under  $T$ . The involution  $T$  acts freely outside  $F_1, F_p$  and the quotient singularities. If we pass to the non-singular model  $Y(p)$  of  $\overline{\mathfrak{S}^2/G}$ , we get the following configuration of curves. We omit the curves coming from the quotient singularities which are pairwise interchanged and only show the intersection behaviour outside of the resolved cusp singularities.



The curves  $F_1, F_p$  are pointwise fixed under the involution  $T$  of  $Y(p)$ , therefore they are non-singular curves on  $Y(p)$ . All curves in the diagram are non-singular and (except  $F_p$ ) rational.  $F_p$  is rational if and only if  $p = 5, 13, 17, 29, 41$  (see 5.7). The points  $P_1$ , and  $P_2$  if  $\varepsilon = 1$ , and  $P_3, P_4$

if  $\delta = 1$  are the only isolated fixed points of  $T$  on  $Y(p)$  outside the resolved cusp singularities.

The following lemma is easy to prove and very useful for deducing from Prestel's results [62] that the configuration on  $Y(p)$  is as indicated in (8).

*Lemma.* *If  $S$  is a compact complex manifold of dimension 2 and  $T$  an involution on  $S$  which carries the non-singular rational curve  $C$  over into itself, then  $T$  is the identity on  $C$  or  $T$  has exactly two fixed points  $P$  and  $Q$  on  $C$ . In the latter case the following holds:*

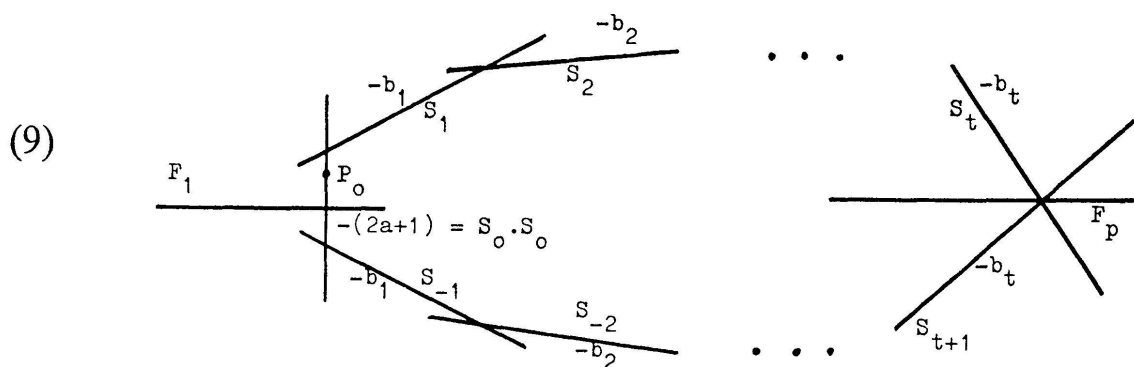
*If  $C \cdot C$  is odd, then one of the points  $P, Q$  is an isolated fixed point of  $T$ , the other one is a transversal intersection point of  $C$  with one of the (non-singular) curves which are pointwise fixed under  $T$ . If  $C \cdot C$  is even, then  $P$  and  $Q$  both are isolated fixed points of  $T$  or both are such transversal intersection points with a curve pointwise fixed under  $T$ .*

The class number  $h$  of  $K = \mathbf{Q}(\sqrt{p})$  is odd. There are  $h$  cusp singularities corresponding to the  $h$  ideal classes (see 3.7). The involution  $T$  on  $\overline{\mathfrak{S}^2/G}$  leaves one cusp fixed and interchanges the others pairwise.  $T$  maps the cusp of type  $(M, U^2)$  where  $M$  is a fractional ideal representing an ideal class to the cusp of type  $(M', U^2)$ . If  $M$  is the  $\mathbf{Z}$ -module  $\mathbf{Z} \cdot w + \mathbf{Z} \cdot 1$  (with  $0 < w' < 1 < w$ ), then  $M'$  is strictly equivalent to  $\mathbf{Z} \frac{1}{w'} + \mathbf{Z} \cdot 1$ .

The resolution of  $(M, U^2)$  is given by the primitive cycle of the purely periodic continued fraction of  $w$ , the resolution of  $(M', U^2)$  by the primitive cycle of  $\frac{1}{w}$ , which is the same cycle in opposite order. The involution on  $Y(p)$  maps the cycle of curves in the resolution of  $(M, U^2)$  onto the cycle of curves in the resolution of  $(M', U^2)$ . The fixed cusp is of type  $(M, U^2)$  where  $M = \mathbf{Z}w_0 + \mathbf{Z} \cdot 1$  and where  $w_0 = \frac{1}{2}(\{\sqrt{p}\} + \sqrt{p})$ , see 4.5 (29). It is the cusp at  $\infty$ .

**THEOREM.** *The length  $r$  of the cycle of  $w_0 = \frac{1}{2}(\{\sqrt{p}\} + \sqrt{p})$  is an odd number  $r = 2t + 1$ . The involution  $T$  on  $Y(p)$  maps the curve  $S_k$  to the curve  $S_{-k}$  (see 4.5). The curve  $F_1$  intersects  $S_0$  transversally. It has the characteristic  $(0|0, 1)$ . The curve  $F_p$  intersects  $S_{-t}$  and  $S_t$ , it has the characteristic  $(-t|1, 1)$ . We put  $\{\sqrt{p}\} = 2a + 1$ . The intersection*

behaviour of the cycle of curves with  $F_1$  and  $F_p$  is illustrated by the following diagram.



$$w_0 = [[2a + 1, b_1, \dots, b_t, b_t, \dots, b_1]]$$

The point  $P_0$  indicates an isolated fixed point of  $T$ . The points  $P_0, P_1$ , and  $P_2$  (if  $\varepsilon = 1$ ), and  $P_3, P_4$  (if  $\delta = 1$ ) are all the isolated fixed points of  $T$ . The curves  $F_1, F_p$  are the only one-dimensional components of the fixed point set.

*Proof.* As in 2.5 and 3.10 we denote ordinary continued fractions by

$$[a_0, a_1, a_2, \dots]. \text{ Then, since } a = \left[ \frac{1 + \sqrt{p}}{2} \right],$$

$$(10) \quad w_0 = \frac{2a + 1 + \sqrt{p}}{2} = [2a, a_1, \dots, a_m, a_m, \dots, a_1, 2a - 1]$$

(See [60], § 30. Because there exists a unit of negative norm, the length of the primitive period in (10) is odd.)

If one applies the formula which transforms the continued fraction (10) into a continued fraction of our type (see 2.5 (19)) one has to go twice over the period in (10). We have

$$(11) \quad w_0 = \left[ \left[ 2a + 1, \underbrace{2, \dots, 2}_{a_1 - 1}, \dots, a_1 + 2, \underbrace{2, \dots, 2}_{2a - 2}, a_1 + 2, \dots, \underbrace{2, \dots, 2}_{a_1 - 1} \right] \right]$$

Thus the length  $r$  of the primitive cycle of  $w_0$  is odd ( $r = 2t + 1$ ). In fact,  $t = a_1 + \dots + a_m + a - 1$ . Under the involution  $T$  only  $S_0$  (self-intersection number  $-(2a + 1)$ ) is carried over into itself. The only symmetric characteristics are  $(0 | 0, 1)$  and  $(-t | 1, 1)$ . The existence of the isolated fixed point  $P_0$  follows from the preceding lemma. Q.E.D.

For the number  $w_0$  in (11) we wish to calculate  $w_{t+k}$  (where  $k = 1, \dots, a$ ), see 4.2. The continued fraction  $[[\dots]]$  of  $w_{t+k}$  begins with  $a - k$  two's. Using again formula 2.5 (19) we obtain

$$0 - \frac{1}{w_{t+k}} = [-1, a - k + 1, a_1, a_2, \dots]$$

$$= -1 + \frac{1}{w_0 - a - k + 1}$$

which yields

$$w_{t+k} = \frac{\sqrt{p} - (2k - 3)}{\sqrt{p} - (2k - 1)} = \frac{M_{t+k} + \sqrt{p}}{2N_{t+k}}$$

where

$$(12) \quad N_{t+k} = \frac{1}{4}(p - (2k - 1)^2), \quad M_{t+k} = 2N_{t+k} + (2k - 1)$$

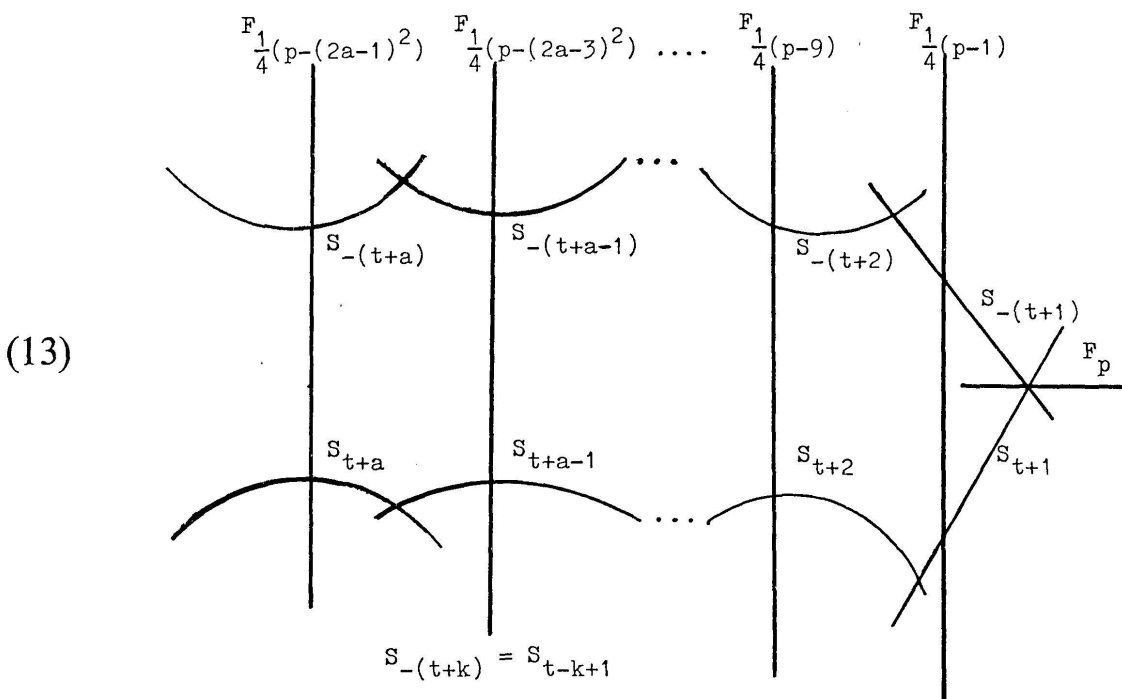
$F_p$  has the characteristic  $(-t | 1, 1) = (t+1 | 1, 1)$  which was obtained in the above proof by a symmetry argument.

It follows also from the theorem in 4.1, because

$$N_{t+1} + N_t + M_t = N_{t+1} + N_{t+1} + M_{t+1} = 4N_{t+1} + 1 = p.$$

In view of (12) and the theorem in 4.1 we have the following proposition.

*Proposition.* On the Hilbert modular surface  $Y(p)$  the cusp at  $\infty$  gives the following configuration of curves  $\left( a = \left[ \frac{1 + \sqrt{p}}{2} \right] \right)$



We have  $S_{t+k} \cdot S_{t+k} = S_{-(t+k)} \cdot S_{-(t+k)} = -2$  for  $1 \leq k \leq a-1$  and  $S_{t+a} \cdot S_{-(t+a)} = -(a_1+2)$ . If  $p = (2a-1)^2 + 4$ , then  $S_{-(t+a)} = S_{t+a} = S_0$ , the curve  $F_{\frac{1}{2}(p-(2a-1)^2)}$  equals  $F_1$  and the diagram has to be changed accordingly. In this case we have

$$w_0 = [2a, \overline{2a-1}] = [[2a+1, \underbrace{2, \dots, 2}_{2a-2}]]$$

and

$$S_{t+a} \cdot S_{t+a} = S_0 \cdot S_0 = -(2a+1).$$

We do not claim that the  $F_N$  are non-singular and do not indicate their mutual intersections nor their intersections with  $F_p$ . The intersections indicated are transversal.

5.5. The curve  $F_1$  on  $Y(p)$  is non-singular. It follows from (8) and 4.3 that it is exceptional. In general, we do not know whether  $F_N$  is non-singular. In view of 4.3 (24) the curves  $F_2, F_3, F_4$  are candidates for exceptional curves. In fact, it follows from Corollaries I, II in 4.4 that they are exceptional if  $Y(p)$  is not rational.  $Y(p)$  is rational if and only if  $p = 5, 13, 17$ . Thus we have

*Lemma.* If  $p$  is a prime  $\equiv 1 \pmod{4}$  and  $> 17$ , then the curves  $F_N$  on the Hilbert modular surface  $Y(p)$  are exceptional for  $N = 1, 2, 3, 4$  provided  $N$  is admissible (see 5.3):

We always have the curve  $F_1$ . The curves  $F_2, F_4$  exist for  $p \equiv 1 \pmod{8}$ . The curve  $F_3$  exists for  $p \equiv 1 \pmod{3}$ .

For the following discussion we assume  $p > 17$ . The curves  $F_1, E, B_1$  in diagram (8) can be blown down successively. In view of corollary III in 4.4, the curves  $F_2, F_3, F_4$  are disjoint and do not intersect any of the curves  $F_1, E, B_1$ . According to the lemma in 5.4 the curves  $F_2, F_3, F_4$  pass through exactly one of the isolated fixed points of the involution  $T$ .

For  $F_3$  the value  $c_1 [F_3]$  equals 1, therefore by 4.3 it meets in  $\mathfrak{H}^2/G$  exactly one quotient singularity of type  $(3; 1, 1)$ , thus it must be the one which is fixed under  $T$ . It intersects  $B_2$  (see (8)) only in  $P_2$  and transversally because otherwise we would have  $c_1 [F_3] > 1$ . The curve  $F_4$  has the model  $\overline{\mathfrak{H}/\Gamma_0(4)}$  which has three cusps. Therefore  $F_4$  must intersect the curves of the resolved cusps of  $\overline{\mathfrak{H}^2/G}$  in three points. One of them is fixed under  $T$ . Thus  $F_4$  passes through  $P_0$ .

The curve  $F_2$  passes through  $P_3$  or  $P_4$  in diagram (8), say  $P_3$ . It intersects  $L$  transversally in  $P_3$  and does not intersect  $L$  in any other point, because otherwise  $L$  would give in the surface with  $F_2$  blown down a curve  $\hat{L}$  with  $c_1[\hat{L}] \geq 2$ . The curves  $F_2, L$  can be blown down successively. Therefore  $L$  is disjoint to any exceptional curve different from  $F_2$ .

We have found an exceptional curve passing through  $P_0$  only for  $p \equiv 1 \pmod 8$ . But there exists such a curve  $F$  for any  $p > 17$ .

For the cusp at  $\infty$  we put as before  $w_0 = \frac{1}{2} (\{\sqrt{p}\} + \sqrt{p}) = \frac{1}{2} (2a+1 + \sqrt{p})$ . The involution  $T$  is given in the coordinate system  $(u_0, v_0)$  by

$$(14) \quad (u_0, v_0) \mapsto (u_0^{-1}, u_0^{-(2a+1)} \cdot v_0),$$

as follows from 2.3 (9). The isolated fixed point  $P_0$  of  $T$  has the coordinates  $(-1, 0)$ . Thus it lies on the curve  $F \subset Y(p)$  given by  $u_0 = -1$  which can be presented in  $\mathfrak{S} \times \mathfrak{S}$  by

$$(15) \quad z_1 = \zeta + \frac{w_0}{2}, \quad z_2 = \zeta + \frac{w_0'}{2}, \quad (\zeta \in \mathfrak{S}).$$

Let  $\Gamma$  be the subgroup of those matrices  $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$  of

$$\begin{pmatrix} 1 & -w_0/2 \\ 0 & 1 \end{pmatrix} \mathbf{SL}_2(\mathfrak{o}_K) \begin{pmatrix} 1 & w_0/2 \\ 0 & 1 \end{pmatrix}$$

which, when acting on  $\mathfrak{S}^2$  carry the diagonal into itself. The curve  $\overline{\mathfrak{S}}/\Gamma$  is a non-singular model of  $F$ . The group  $\Gamma$  is characterized by 4.1 (1), but the second condition is impossible. Thus  $\Gamma$  is the subgroup of  $\mathbf{SL}_2(\mathbf{Q})$  of matrices  $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$  for which

$$\begin{pmatrix} \alpha + \gamma w_0/2 & -\alpha w_0/2 + \beta - \gamma w_0^2/4 + \delta w_0/2 \\ \gamma & \delta - \gamma w_0/2 \end{pmatrix}$$

is integral. Since  $w_0, 1$  is a  $\mathbf{Z}$ -base of  $\mathfrak{o}_K$ , we get that  $\alpha, \delta$  are integers and  $\gamma$  is an even integer. We have

$$(16) \quad -\alpha w_0/2 + \beta - \gamma w_0^2/4 + \delta w_0/2 = \\ (-\alpha/2 - \gamma(2a+1)/4 + \delta/2) w_0 + \beta + \gamma((2a+1)^2 - p)/16$$

If  $p \equiv 1 \pmod 8$ , then  $\beta$  is an integer and  $\alpha\delta - \beta\gamma = 1$  implies  $\alpha \equiv \delta \pmod 2$  and  $\gamma \equiv 0 \pmod 4$ , because the coefficient of  $w_0$  in (16) must be integral. Thus  $\Gamma = \Gamma_0(4)$  in this case.

If  $p \equiv 5 \pmod{8}$ , then  $\Gamma_0(4) \subset \Gamma$ . We put  $\gamma = 2\gamma^*$  and  $\beta = \beta^*/2$ . Then  $\gamma^*, \beta^*$  are integers which are congruent modulo 2. We have  $\alpha + \delta \equiv \gamma^* \pmod{2}$ .

The matrix  $\begin{pmatrix} 1 & -\frac{1}{2} \\ 2 & 0 \end{pmatrix}$ , whose third power is  $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ , satisfies these conditions.  $\Gamma$  is a normal extension of index 3 of  $\Gamma_0(4)$ . The three cusp of  $\mathfrak{H}/\Gamma_0(4)$  are identified.  $\overline{\mathfrak{H}/\Gamma}$  is a rational curve. Put  $\tilde{\Gamma} = \Gamma/\{1, -1\}$ . We have  $a_3(\tilde{\Gamma}) = 2$  ( $a_r(\tilde{\Gamma}) = 0$  otherwise) and  $\sigma(\tilde{\Gamma}) = 1$ .

Therefore  $c_1(\tilde{\Gamma}) = 1$  (see the definition in 4.3), and the curve  $F$  is exceptional. It passes through the isolated fixed point  $P_0$  of  $T$ . For  $p \equiv 1 \pmod{8}$ , the curve  $F$  equals  $F_4$  because two different exceptional curves do not intersect. We have  $T(F) = F$ .

We can now state the following proposition.

*Proposition.* *If we blow down the curves  $F_1, E, B_1, F$ , and  $F_2, L$  (for  $\delta = 1$ ), and  $F_3$  (for  $\varepsilon = 1$ ) on the surface  $Y(p)$  for  $p > 17$ , then we obtain a non-singular algebraic surface  $Y^0(p)$ . The involution  $T$  is also defined on  $Y^0(p)$ . It does not have any isolated fixed point. The curve  $F_p$  has a non-singular image  $F_p^0$  in  $Y^0(p)$  which is the complete fixed point set of  $T$ .*

5.6. If  $c_1$  is again the first Chern class of  $Y(p)$ , then

$$(17) \quad c_1[F_p] = -\frac{p+1}{6} + \frac{\varepsilon}{3} + 2$$

This follows from 4.3 (19), because  $[\mathbf{SL}_2(\mathbf{Z}) : \Gamma_0(p)] = p + 1$  and  $[\Gamma^*(p) : \Gamma_0(p)] = 2$ . We further use (8) and (9).

Let us now assume that  $Y(p)$  is not rational which is the case for  $p > 17$ . In  $Y(p)$  we have blown down  $3 + 1 + 2\delta + \varepsilon$  curves and obtained the surface  $Y^0(p)$  on which  $T$  has the fixed point set  $F_p^0$ . Let  $c_1^0$  be the first Chern class of  $Y^0(p)$ . Then

$$(18) \quad c_1^0[F_p^0] = -\frac{p+1}{6} + \frac{\varepsilon}{3} + 2 + 2 + 1 + 2\delta + \varepsilon.$$

This follows from 4.4 (25a) using that  $F, F_2, F_3$  intersect  $F_p$  transversally in exactly one point (see the lemma in 5.4). By 5.1 (1) the number  $c_1^0[F_p^0]$  must be divisible by 4. We have

$$(19) \quad \frac{1}{4}c_1^0[F_p^0] = -\left[\frac{p-29}{24}\right],$$



since  $\frac{1}{4} \left( \frac{\varepsilon}{3} + 2\delta + \varepsilon \right) < 1$ . The surface  $Y^0(p)/T$  is a non singular model for the compactification of  $\mathfrak{S}^2/G_T$  (see 5.2). The arithmetic genus of  $Y^0(p)/T$  will be denoted by  $\chi_T(p)$ . In 3.12 we have given a formula for the arithmetic genus of  $Y(p)$  which we shall call here  $\chi(p)$ . Then

$$(20) \quad \chi(p) = \frac{1}{2} \zeta_K(-1) + \frac{h(-4p)}{8} + \frac{1}{6} h(-3p),$$

where  $K = \mathbf{Q}(\sqrt{p})$ . By 5.1 (1) and (19) the arithmetic genera  $\chi(p)$  and  $\chi_T(p)$  are related by the formula [40]

$$(21) \quad \chi_T(p) = \frac{1}{2} \left( \chi(p) - \left[ \frac{p-29}{24} \right] \right),$$

(compare [14], Part II, Satz 2).

This formula is also valid for  $p = 5, 13, 17$ . In these cases the surface  $Y(p)$  and therefore also  $Y^0(p)/T$  are rational and (21) reduces to  $1 = \frac{1}{2}(1+1)$ . It was shown in [40] that

$$\chi_T(p) > \frac{p^{3/2}}{1440} - \frac{p+1}{48} \quad (\text{compare 3.12}),$$

and explicit calculations gave the result that  $\chi_T(p) = 1$  for exactly 24 primes, namely for all primes ( $\equiv 1 \pmod{4}$ ) smaller than the prime 193 and for  $p = 197, 229, 269, 293, 317$ .

*We wish to show in the next sections that the surfaces  $Y^0(p)/T$  are rational for these primes. Since the rationality is already known for  $p = 5, 13, 17$  it remains to consider 21 primes. Since the first Betti number of  $Y(p)$  vanishes (3.6), the same holds for  $Y^0(p)/T$ . Thus the rationality criteria of 4.4 (Corollaries I, II, III) can be applied.*

5.7. The curve  $F_N$  in  $Y(p)$  (for an admissible natural number  $N > 4$ ) projects down to a curve  $F_N^0$  in  $Y^0(p)$  and to a curve  $F_N^* = F_N^0/T$  in  $Y^0(p)/T$ . If  $N$  is not divisible by  $p$ , then  $F_N^*$  has  $\overline{\mathfrak{S}/\Gamma^*(N)}$  as non-singular model (see the remark in 5.3). We have a commutative diagram:

$$\begin{array}{ccc} \overline{H / \Gamma_0(N)} & \rightarrow & F_N^0 \subset Y_0(p) \\ \downarrow & & \downarrow \quad \downarrow \\ \overline{H / \Gamma_*(N)} & \rightarrow & F_N^* \subset Y_0(p) / T \end{array}$$

There is an involution  $\tau$  on  $\overline{\mathfrak{H}/\Gamma_0(N)}$  compatible with  $T$  and having  $\overline{\mathfrak{H}/\Gamma_*(N)}$  as orbit space. Recall that  $F_p^0$  is the fixed point set of  $T$  on  $Y^0(p)$ . Thus the intersection number  $F_N^0 \cdot F_p^0$  is greater or equal to the number  $\text{fix}(\tau)$  of fixed points of  $\tau$  on  $\overline{\mathfrak{H}/\Gamma_0(N)}$ :

$$(22) \quad F_N^0 \cdot F_p^0 \geq \text{fix}(\tau) = 2e(\overline{\mathfrak{H}/\Gamma_*(N)}) - e(\overline{\mathfrak{H}/\Gamma_0(N)})$$

Let  $c_1^*$  be the first Chern class of  $Y^0(p)/T$ . By 5.1 (2) we get

$$c_1^*[F_N] = \frac{1}{2}(c_1^0[F_N^0] + F_N^0 \cdot F_p^0)$$

Since  $c_1^0[F_N^0] \geq c_1[F_N] \geq c_1(N)$ , see 4.3 and 4.4 (25a), the following estimate is obtained:

$$(23) \quad c_1^*[F_N^*] \geq \frac{1}{2}c_1(N) + e(\overline{H/\Gamma_*(N)}) - \frac{1}{2}e(\overline{H/\Gamma_0(N)})$$

The right side of (23) only depends on  $N$ . We shall denote it by  $c_1^*(N)$  and have

$$(24) \quad c_1^*[F_N^*] \geq c_1^*(N).$$

There are explicit formulas for the Euler numbers or equivalently the genera of the curves  $\overline{\mathfrak{H}/\Gamma_*(N)}$ , see [16], p. 357, and [13]. Helling [32] has shown that there are exactly 37 values  $N \geq 2$  for which  $\overline{\mathfrak{H}/\Gamma_*(N)}$  is rational. (In [16], p. 367, Fricke omits the value  $N = 59$ ). We shall give a list of the  $c_1^*(N)$  for the 34 values  $\geq 5$ .

By the definition of  $c_1(N)$  we get:

If  $\overline{\mathfrak{H}/\Gamma_*(N)}$  is rational, then (for  $N \geq 5$ )

$$(25) \quad c_1^*(N) = 3 - g_0(N) - \frac{1}{2}(a_2(N) + a_3(N) + \sigma_0(N)).$$

Using [13] we obtain the following list:

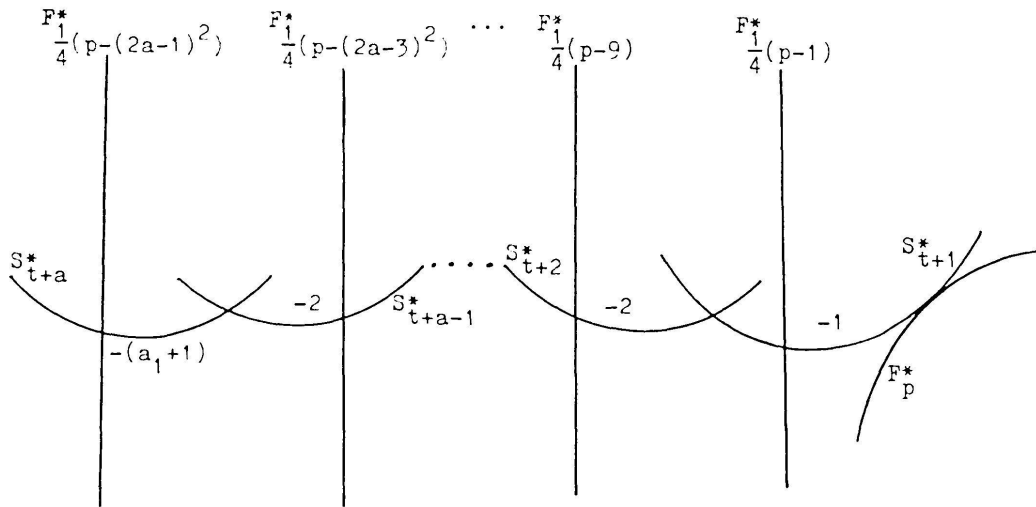
$$e(\overline{\mathfrak{H}/\Gamma_*(N)}) = 2$$

$N$	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
$c_1^*(N)$	1	1	1	1	1	0	1	0	0	0	0	0	0	-1	0	-1	-1

$N$	23	24	25	26	27	29	31	32	35	36	39	41	47	49	50	59	71
$c_1^*(N)$	0	-2	-1	-2	-1	-1	-1	-2	-2	-4	-3	-2	-2	-3	-6	-3	-4

5.8. The curves  $F_N$  will be used for rationality proofs. Consider the diagram (13) for  $p > 17$ . We have  $\frac{1}{4}(p - (2a - 3)^2) \geq 5$ . It follows from 4.2 (15) that the exceptional curves  $F_1, F, F_2, F_3$  do not intersect  $S_{t+k}$  and  $S_{-(t+k)}$  for  $1 \leq k \leq a - 1$ . These exceptional curves also do not meet  $S_{t+a}$  and  $S_{-(t+a)}$  if  $\frac{1}{4}(p - (2a - 1)^2) \geq 5$ . In this case, the configuration (13) is not changed by passing to  $Y^0(p)$ . If we apply the involution  $T$  we get the following configuration on  $Y^0(p)/T$ .

(27)



If  $\frac{1}{4}(p - (2a - 1)^2) < 5$ , the diagram has to be changed. But the sub-diagram of (27) obtained by not showing  $F^*_{\frac{1}{4}(p - (2a - 1)^2)}$  and  $S^*_{t+a}$  exists on the surface  $Y^0(p)/T$  for any  $p > 17$ .

We do not know whether the curves  $F^*_{\frac{1}{4}(p - (2k - 1)^2)}$  are non-singular and do not claim anything about their mutual intersection behaviour. The  $S^*_{t+k}$  are the image of  $S_{t+k}$  and  $S_{-(t+k)}$ . They are non-singular. The equation  $S^*_{t+1} \cdot S^*_{t+1} = -1$  or equivalently  $c^*_1[S^*_{t+1}] = 1$  follows from 5.1 (2). The curves  $S^*_{t+k}$  ( $1 \leq k \leq a - 1$ ) can be blown down successively. Then  $F^*_{(p - (2k - 1)^2)/4}$  gives in the resulting surface a curve for which the value of the first Chern class of the new surface on this curve is greater or equal to  $c^*_1((p - (2k - 1)^2)/4) + a - k$ .

*Proposition.* Let  $p$  be a prime  $\equiv 1 \pmod{4}$  (and  $p > 17$ ). The non-singular model  $Y^0(p)/T$  for the symmetric Hilbert modular group is rational

if there exists a natural number  $k$  with  $1 \leq k \leq a - 1 = \left\lfloor \frac{\sqrt{p} - 1}{2} \right\rfloor$  such

that

$$c^*_1((p - (2k - 1)^2)/4) + a - k \geq 2$$

This is a consequence of corollary I in 4.4. For the above proposition one does not need any assumption about the genus of  $F_N$  where  $N = \frac{1}{4}(p - (2k-1)^2)$ . However, we shall try to get through using the  $N$  listed in 5.7 for which the curves  $F_N$  are rational.

The tables in 5.7 give immediately

$$c_1^* \left( \frac{p-1}{4} \right) + a - 1 \geq 2$$

for  $p = 29, 37, 41, 53, 61, 73, 97, 101, 109, 197$ .

We find

$$c_1^* \left( \frac{p-9}{4} \right) + a - 2 \geq 2$$

for  $p = 89, 137, 293$ .

For  $p = 173$  we have

$$c_1^* \left( \frac{p-81}{4} \right) + a - 5 = c_1^*(23) + 7 - 5 = 2$$

For the remaining 7 primes 113, 149, 157, 181, 229, 269, 317 we shall try to use the following lemma.

*Lemma.* We keep the notations of the preceding proposition. Suppose there exist two natural numbers  $k_1, k_2$  with  $1 \leq k_1 < k_2 \leq a-1$  such that

$$c_1^* \left( \frac{p - (2k_i - 1)^2}{4} \right) + a - k_i = 1 \quad \text{for } i = 1, 2$$

Then  $Y^0(p)/T$  is rational.

*Proof.* Blowing down  $S_{i+1}^*, \dots, S_{i+a-1}^*$  in  $Y^0(p)/T$  gives a surface in which the images of  $F_{N_i}^*$  ( $N_i = (p - (2k_i - 1)^2)/4$ ,  $i = 1, 2$ ) are exceptional curves or the surface is rational (4.4, Corollary II). If we have the two exceptional curves, then they intersect and the surface is rational by Corollary III in 4.4.

The assumptions of the lemma are true for  $p = 113$  and  $k_1 = 2$ ,  $k_2 = 4$ , for  $p = 149$  and  $k_1 = 4$ ,  $k_2 = 5$ , for  $p = 157$  and  $k_1 = 4$  and  $k_2 = 5$ , for  $p = 181$  and  $k_1 = 5$ ,  $k_2 = 6$ , for  $p = 229$  and  $k_1 = 6$ ,  $k_2 = 7$ , for  $p = 317$  and  $k_1 = 5$ ,  $k_2 = 8$ .

For  $p = 269$  we have  $a = 8$ . The curve  $S_{t+8}^*$  has self-intersection number  $-3$ . It intersects  $F_{11}^*$ , since  $11 = \frac{269 - 15^2}{4}$ . Either the surface is rational or  $F_{11}^*$  is exceptional. If  $F_{11}^*$  is exceptional, then we blow down  $F_{11}^*, S_{t+1}^*, \dots, S_{t+8}^*$ . The curve  $F_{47}^*$  ( $k = 5$ ) gives in the resulting surface  $\tilde{Y}$  a curve  $\tilde{D}$  with  $\tilde{c}_1 [D] \geq 2$  where  $\tilde{c}_1$  is the first Chern class of  $\tilde{Y}$ .

We have proved the desired result.

**THEOREM.** *Let  $p$  be a prime  $\equiv a \pmod{4}$ . Let  $G_T$  be the symmetric Hilbert modular group for  $K = \mathbf{Q}(\sqrt{p})$ . Then the surface  $\overline{\mathfrak{H}^2/G_T}$  is rational, (or equivalently the field of meromorphic automorphic functions with respect to  $G_T$  is a purely transcendental extension of  $\mathbf{C}$ ), if and only if  $p < 193$  or  $p = 197, 229, 269, 293, 317$ .*

5.9. *Example.* If the prime  $p \equiv 1 \pmod{4}$  is of the form

$$p = (2a - 1)^2 + 4,$$

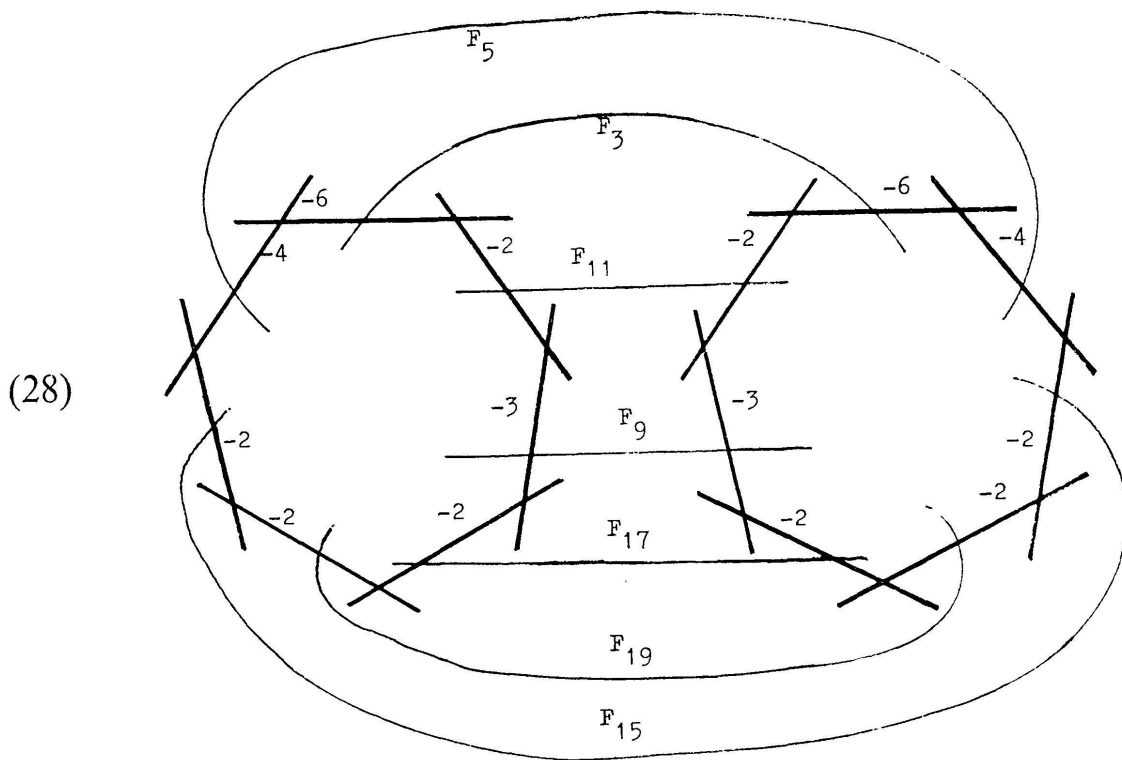
then 
$$w_0 = \frac{2a + 1 + \sqrt{p}}{2} = \underbrace{[[2a + 1, 2, \dots, 2]]}_{2a-2}$$

and we have in diagram (13) that  $S_{t+a} = S_{-(t+a)} = S_0$ . Since  $(p - (2a - 3)^2)/4 = 2a - 1$ , the smallest admissible  $N > 1$  which can be written in the form  $x^2 N_k + xy M_k + y^2 N_{k-1}$  (with integers  $x, y \geq 0$ ) equals  $2a - 1$  (see 4.2 and 5.4 (12)). Any divisor  $d$  of  $2a - 1$  is admissible. If  $d$  is a prime dividing  $2a - 1$  and  $1 < d < 2a - 1$ , then the curve  $F_d$  has two cusps and does not pass through the cusp at  $\infty$  of  $\overline{\mathfrak{H}^2/G}$ . Thus there must be other cusps of  $\overline{\mathfrak{H}^2/G}$ . We have proved

*Proposition.* *If  $p = (2a - 1)^2 + 4$  ( $p$  prime) and if  $2a - 1$  is not a prime, then  $h(p) > 1$ . (See [29], [51]).*

The first example is  $p = 229 = 15^2 + 4$ . We have  $h(p) = 3$ . The number 229 is the only one of our 24 primes in the preceding theorem with class number greater than one. (If  $(2a - 1) \equiv \pm 2 \pmod{7}$ , then 7 is admissible for  $p$ . Thus, also in this case  $h(p) > 1$  provided  $2a - 1 > 7$ . Example:  $p = 1373 = 37^2 + 4$ ,  $h(p) = 3$ .)

The cycles for the 2 cusps not at  $\infty$  of  $Y(229)$  look as follows



We also have drawn some curves. The curve  $F_{15}$  has  $\overline{\mathfrak{H}}/\Gamma_0(15)$  as non-singular model. This has 4 cusps corresponding to the fact that  $F_{15}$  also passes through the cusp at  $\infty$  of  $Y(p)$ , namely through the curves  $S_1$  and  $S_{-1}$  of this cusp. One can show that  $F_9$  passes through  $S_0$  of the cusp at  $\infty$  in two points ( $\overline{\mathfrak{H}}/\Gamma_0(9)$  has 4 cusps).

If  $N$  is admissible and is a product of  $k$  different primes ( $\neq p$ ), then  $\overline{\mathfrak{H}}/\Gamma_0(N)$  has  $2^k$  cusps. The  $2^k$  intersections of  $F_N$  with the resolved cusps in  $Y(p)$  correspond to  $2^k$  admissible ideals  $\mathfrak{b}$  with  $N(\mathfrak{b}) = N$  (see 5.3).

In general, it is possible to give a complete description of the intersection of  $F_N$  with the resolved cusps of  $Y(p)$ . The corresponding theory can be developed for any Hilbert modular surface.

ADDED IN PROOF:

A. Selberg has informed me that he has proved the following result.

If  $\Gamma$  is a discrete irreducible subgroup of  $(\mathbf{PL}_2^+(\mathbf{R}))^n$  such that  $\mathfrak{H}^n/\Gamma$  has finite volume, but is not compact, then  $\Gamma$  is conjugate in  $(\mathbf{PL}_2^+(\mathbf{R}))^n$  to a group commensurable with the Hilbert modular group of some totally real field  $K$  with  $[K:\mathbf{Q}] = n$ .

Thus Selberg's conjecture mentioned in the remark at the end of 1.5 is true. Actually, Selberg's results are more general. The proof has not been published yet. There is a sketch (still involving additional assumptions which could be eliminated later) in the Proceedings of the 15th Scandinavian

congress, Oslo 1968, Lecture Notes in Mathematics, Springer Verlag, vol. 118; in particular pp. 106-113.

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