2. Boundedness and convergence preserving regular operators.

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V_Ψ is additive on non-negative functions in $\mathscr{F}_1$ and homogeneous with respect to multiplication by positive real numbers:

$$f_1 \geqq 0 \text{ and } f_2 \geqq 0 \Rightarrow V_\Psi(f_1+f_2, x) = V_\Psi(f_1, x) + V_\Psi(f_2, x),$$
$$\alpha > 0 \text{ and } f \geqq 0 \Rightarrow V_\Psi(\alpha f, x) = \alpha V_\Psi(f, x).$$

The operator V_Ψ is then extended to a positive linear operator $\Phi_\Psi\colon \mathscr{F}_1 \to \mathscr{F}$, which coincides with V_Ψ on non-negative functions, in the usual way: if $f = f^+ - f^-$, then $\Phi_\Psi(f, x) = V_\Psi(f^+, x) - V_\Psi(f^-, x)$. Finally, if $f \in \mathscr{F}_1$ we have $|\Psi(f, x)| \leqq V_\Psi(|f|, x) = \Phi_\Psi(|f|, x)$. Hence, by Theorem I, the operator Ψ is regular.

Conversely, if Ψ is regular, by Theorem I, we have a positive linear operator $\Phi\colon \mathscr{F}_1 \to \mathscr{F}$ such that

$$|\Psi(g, x)| \leqq \Phi(|g|, x) \leqq \Phi(f, x)$$

for every $g \in \mathscr{F}_1$, $|g| \leqq f$, and every $x \in R^+$. Hence, the condition (1.4) of Theorem II is satisfied.

If we consider the operator G defined by (1.1), then

$$V_G(f, x) = \int_0^\infty |\psi(x, t)| f(t)\, dt.$$

From the statement of the theorems of Hahn and Raff mentioned earlier, we can expect that the operator V_Ψ will play an important role in the extension of these results to general regular operators. In fact, as in the theory of positive linear operators, some asymptotic property of the regular operator Ψ will hold for a large class of functions if and only if the operator V_Ψ has certain properties on a much smaller class of functions.

2. Boundedness and convergence preserving regular operators.

In this section and the following one we shall extend to regular operators some of the well-known results about the asymptotic behavior of the special transform G defined by (1.1).

Let us consider the linear space $\mathscr{M}$ of real valued measurable functions on R^+ and let $\mathscr{M}_0$ be the subspace of $\mathscr{M}$ consisting of all measurable functions on R^+ which are bounded on every finite interval of R^+.

The basic result which characterizes regular operators from $\mathscr{M}_0$ into $\mathscr{F}_0$ that preserve boundedness can be stated as follows:

THEOREM 1. *Let* $\Psi : \mathcal{M}_0 \to \mathcal{F}_0$ *be a regular operator. In order that, as* $x \to \infty$,

$$f \in \mathcal{M}_0 \text{ and } f(x) = O(1) \quad \Rightarrow \quad \Psi(f, x) = O(1) \tag{2.1}$$

it is necessary and sufficient that

$$V_\Psi(1, x) = O(1) \tag{2.2}$$

where V_Ψ *is defined by* (1.5).

This result is clearly a natural extension of the results for the special operator G mentioned in section *1.2* under *A*. The corresponding result for regular operators which transform functions in $\mathcal{S}_0$, converging to zero as $x \to \infty$ into bounded functions is given by the following theorem.

THEOREM 2. *Let* $\Psi : \mathcal{M}_0 \to \mathcal{F}_0$ *be a regular operator. In order that, as* $x \to \infty$,

$$f \in \mathcal{M}_0 \text{ and } f(x) \to 0 \quad \Rightarrow \quad \Psi(f, x) = O(1) \tag{2.3}$$

it is necessary and sufficient that

$$W_\Psi(1, x) = O(1) \tag{2.4}$$

where W_Ψ *is defined for every* $f \in \mathcal{M}_0, f \geqq 0$, *by*

$$W_\Psi(f, x) = \sup \{ | \Psi(g, x) | : g \in \mathcal{M}_0, | g | \leqq f, g = o(f) \}^{1)} . \tag{2.5}$$

Condition (2.4) in Theorem 2 is less restrictive than the corresponding condition (2.2) in Theorem 1, since $W_\Psi(f, x) \leqq V_\Psi(f, x)$ for every $f \in \mathcal{M}_0$, $f \geqq 0$ and for every $x \in R^+$.

It is now easy to obtain a generalization of the results for the special operator G mentioned in section *1.2* under *B.*, i.e. to establish necessary and sufficient conditions for a regular operator to be convergence preserving:

THEOREM 3. *Let* $\Psi : \mathcal{M}_0 \to \mathcal{F}_0$ *be a regular operator. In order that, as* $x \to \infty$,

$$f \in \mathcal{M}_0 \text{ and } f(x) \to c \quad \Rightarrow \quad \Psi(f, x) \to c \tag{2.6}$$

it is necessary and sufficient that

$$\Psi(1, x) \to 1 , \tag{2.7}$$

[1] $g = o(f)$ means $g(t) = o(f(t))$ $(t \to \infty)$.

$$\Psi(\chi_E, x) \to 0 \tag{2.8}$$

for every bounded measurable subset E of R^+, and

$$W_\Psi(1, x) = O(1). \tag{2.9}$$

3. Transformations of O-regular and slowly varying functions by regular operators.

3.1. The class of positive functions which are eventually bounded away from zero and infinity has been extended to the class of O-regular functions defined as follows:

A positive, measurable function l on R^+ is O-regular if

$$\frac{l(\lambda x)}{l(x)} = O(1) \quad (x \to \infty) \tag{3.1}$$

for every $\lambda > 0$.

For example, any function l such that $ax^\alpha \leqq l(x) \leqq Ax^\alpha$, where $\alpha \in R$, clearly satisfies condition (3.1).

The class of O-regular functions and related classes of functions have been studied extensively by V. G. Avakumović [8, 9, 10, 11], J. Karamata [14], N. K. Bari, S. B. Stečkin [15], M. A. Krasnoselskiĭ, T. B. Rutickiĭ [16], W. Matuszewska [17] and others.

The closely related class of slowly varying (SV) functions, introduced by J. Karamata ([12], [13]), generalizes the class of functions converging to a positive limit. A positive, measurable function L defined on R^+ is a slowly varying function if

$$\lim_{x \to \infty} \frac{L(\lambda x)}{L(x)} = 1 \tag{3.2}$$

for every $\lambda > 0$.

Clearly, every measurable function on R^+ which converges to a positive limit as $x \to \infty$ is a SV function. Also, functions like

$$\varphi(x) = \begin{cases} 1, 0 \leqq x < e, \\ \log x, x \geqq e, \end{cases} \quad h(x) = \left(2 + \frac{\sin x}{x}\right) \varphi(x),$$

and their iterations are SV functions. More generally, any measurable function g on R^+ such that $\varphi(x) \leqq g(x) \leqq \varphi(x) + \sqrt{\varphi(x)}$ is a SV function.

The most important properties of O-regular and SV functions can be stated as follows: