

### 3. Transformations of 0-regular and slowly varying functions by regular operators.

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **19 (1973)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **11.07.2024**

#### **Nutzungsbedingungen**

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern. Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden. Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

#### **Haftungsausschluss**

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

$$(2.8) \quad \Psi(\chi_E, x) \rightarrow 0$$

for every bounded measurable subset  $E$  of  $R^+$ , and

$$(2.9) \quad W_\Psi(1, x) = O(1).$$

### 3. TRANSFORMATIONS OF $O$ -REGULAR AND SLOWLY VARYING FUNCTIONS BY REGULAR OPERATORS.

3.1. The class of positive functions which are eventually bounded away from zero and infinity has been extended to the class of  $O$ -regular functions defined as follows:

A positive, measurable function  $l$  on  $R^+$  is  $O$ -regular if

$$(3.1) \quad \frac{l(\lambda x)}{l(x)} = O(1) \quad (x \rightarrow \infty)$$

for every  $\lambda > 0$ .

For example, any function  $l$  such that  $ax^\alpha \leq l(x) \leq Ax^\alpha$ , where  $\alpha \in R$ , clearly satisfies condition (3.1).

The class of  $O$ -regular functions and related classes of functions have been studied extensively by V. G. Avakumović [8, 9, 10, 11], J. Karamata [14], N. K. Bari, S. B. Stečkin [15], M. A. Krasnoselskiĭ, T. B. Rutickiĭ [16], W. Matuszewska [17] and others.

The closely related class of slowly varying ( $SV$ ) functions, introduced by J. Karamata ([12], [13]), generalizes the class of functions converging to a positive limit. A positive, measurable function  $L$  defined on  $R^+$  is a slowly varying function if

$$(3.2) \quad \lim_{x \rightarrow \infty} \frac{L(\lambda x)}{L(x)} = 1$$

for every  $\lambda > 0$ .

Clearly, every measurable function on  $R^+$  which converges to a positive limit as  $x \rightarrow \infty$  is a  $SV$  function. Also, functions like

$$\varphi(x) = \begin{cases} 1, & 0 \leq x < e, \\ \log x, & x \geq e, \end{cases}, \quad h(x) = \left(2 + \frac{\sin x}{x}\right) \varphi(x),$$

and their iterations are  $SV$  functions. More generally, any measurable function  $g$  on  $R^+$  such that  $\varphi(x) \leq g(x) \leq \varphi(x) + \sqrt{\varphi(x)}$  is a  $SV$  function.

The most important properties of  $O$ -regular and  $SV$  functions can be stated as follows:

REPRESENTATION THEOREMS: *If  $l$  is an  $O$ -regular function, there exist  $B > 0$  and bounded measurable functions  $\alpha$  and  $\beta$  on  $[B, \infty]$  such that*

$$(3.3) \quad l(x) = \exp \left( \alpha(x) + \int_B^x \frac{\beta(t)}{t} dt \right) \text{ for } x \geq B.$$

*If  $L$  is a  $SV$  function, then for some  $B > 0$ ,*

$$(3.4) \quad L(x) = \exp \left( \eta(x) + \int_B^x \frac{\varepsilon(t)}{t} dt \right) \text{ for } x \geq B,$$

*where  $\eta$  and  $\varepsilon$  are bounded measurable functions on  $[B, \infty]$  such that  $\eta(x) \rightarrow c$  and  $\varepsilon(x) \rightarrow 0$  ( $x \rightarrow \infty$ ).*

A proof of these results for continuous  $O$ -regular and  $SV$  functions can be found in [12], [13], and [14]. These results were subsequently extended to measurable  $O$ -regular and  $SV$  functions by a number of authors (see [18] for details).

One of the typical and simplest results about the asymptotic behavior of special linear transforms of  $SV$  functions is probably the following result of K. Knopp [19]:

If  $L$  is a  $SV$  function, and if  $L \in \mathcal{M}_0$ , then

$$\frac{1}{xL(x)} \int_0^\infty e^{-(t/x)} L(t) dt \rightarrow 1 \quad (x \rightarrow \infty).$$

Similar results involving more or less special transformations have been obtained by G. H. Hardy and W. W. Rogosinski [4], S. Aljančić, R. Bojanić, M. Tomić [20], R. Bojanić and J. Karamata [21], and, in slightly different form, by D. Drasin ([22], Th. 6). The most general result of this type, obtained by M. Vuilleumier [23], [24], can be stated as follows:

*Let  $G$  be defined by (1.1). In order that*

$$\frac{G(L, x)}{L(x)} \rightarrow 1 \quad (x \rightarrow \infty)$$

*holds for every  $SV$  function  $L \in \mathcal{M}_0$  it is necessary and sufficient that, as  $x \rightarrow \infty$ ,*

$$(i) \quad \int_0^\infty \Psi(x, t) dt \rightarrow 1,$$

(ii) *there exists  $\eta > 0$  such that*

$$\int_0^x |\Psi(x, t)| t^{-\eta} dt = O(x^{-\eta}) \text{ and } \int_x^\infty |\Psi(x, t)| t^\eta dt = O(x^\eta).$$

3.2. Theorem 1 characterizes boundedness preserving operators. A natural extension of that result is the theorem which characterizes regular operators  $\Psi$  with the property that  $\Psi(l, x) = O(l(x))$  ( $x \rightarrow \infty$ ) holds for every  $O$ -regular function  $l \in \mathcal{M}_0$ . In this direction we have the following result:

THEOREM 4. *Let  $\Psi: \mathcal{M}_0 \rightarrow \mathcal{F}_0$  be a regular operator. In order that*

$$(3.5) \quad \Psi(l, x) = O(l(x)) \quad (x \rightarrow \infty),$$

*holds for every  $O$ -regular function  $l \in \mathcal{M}_0$  it is necessary and sufficient that for all  $\alpha > 0$ , as  $x \rightarrow \infty$ ,*

$$(3.6) \quad V_\Psi(t^\alpha, x) = O(x^\alpha)$$

*and*

$$(3.7) \quad V_\Psi(\chi_{[0,1]}(t) + t^{-\alpha} \chi_{(1,\infty)}(t), x) = O(x^{-\alpha})$$

*where  $V_\Psi$  is defined by (1.5).*

Likewise, as an analog of Theorem 2, the following theorem characterizes regular operators which have the property that

$$\Psi(L, x) = O(L(x)) \quad (x \rightarrow \infty)$$

holds for every  $SV$  function  $L \in \mathcal{M}_0$ :

THEOREM 5. *Let  $\Psi: \mathcal{M}_0 \rightarrow \mathcal{F}_0$  be a regular operator. In order that*

$$(3.8) \quad \Psi(L, x) = O(L(x)) \quad (x \rightarrow \infty)$$

*holds for every  $SV$  function  $L \in \mathcal{M}_0$  it is necessary and sufficient that there exists  $\eta > 0$  such that, as  $x \rightarrow \infty$ ,*

$$(3.9) \quad W_\Psi(t^\eta, x) = O(x^\eta)$$

*and*

$$(3.10) \quad W_\Psi(\chi_{[0,1]}(t) + t^{-\eta} \chi_{(1,\infty)}(t), x) = O(x^{-\eta})$$

*where  $W_\Psi$  is defined by (2.5).*

Finally, the analog of Theorem 3 can be stated as follows:

THEOREM 6. Let  $\Psi: \mathcal{M}_0 \rightarrow \mathcal{F}_0$  be a regular operator. In order that

$$(3.11) \quad \frac{\Psi(L, x)}{L(x)} \rightarrow 1 \quad (x \rightarrow \infty)$$

holds for every SV function  $L \in \mathcal{M}_0$  it is necessary and sufficient that

$$(3.12) \quad \Psi(1, x) \rightarrow 1 \quad (x \rightarrow \infty),$$

and that the asymptotic relations (3.9) and (3.10) hold for some  $\eta > 0$ .

#### 4. PROOFS.

4.1. Proof of Theorem 1. The sufficiency of condition (2.2) follows from the inequality

$$|\Psi(f, x)| \leq V_\Psi(1, x) \|f\|.$$

The necessity of (2.2) is proved by way of contradiction. Suppose that (2.2) is not satisfied. Then

$$(4.1.1) \quad \limsup_{x \rightarrow \infty} V_\Psi(1, x) = \infty.$$

In view of (4.1.1), (2.1) and the properties of  $\Psi$ , it is possible to find by induction an increasing sequence  $(x_k)$  going to infinity and a sequence  $(g_k)$  of functions in  $\mathcal{M}_0$  such that, if  $A_k$  is defined by  $A_k = V_\Psi(1, x_k)$ , then

$$(4.1.2) \quad A_1 \geq 16 \text{ and } A_k \geq 16 A_{k-1}, \quad k = 2, 3, \dots,$$

$$(4.1.3) \quad A_k \geq 16 \left( \sup_{x \in R^+} |\Psi \left( \sum_{i=1}^{k-1} \frac{g_i}{\sqrt{A_i}}, x \right)| \right)^2, \quad k = 2, 3, \dots,$$

and

$$(4.1.4) \quad |g_k| \leq 1, \quad |\Psi(g_k, x_k)| \geq \frac{3}{4} A_k, \quad k = 1, 2, \dots$$

Let

$$(4.1.5) \quad g(x) = \sum_{i=1}^{\infty} \frac{g_i(x)}{\sqrt{A_i}}.$$