AN INTEGRAL INEQUALITY IN ANALYTIC FUNCTION THEORY

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The following theorem was proved in [1]:

THEOREM A. *Suppose that $f = f(z)$ is an entire function of a complex variable z. Then the only solutions of the functional inequality*

$$(1) \qquad |f((x+y)/2)| \leqq (|f(x)| + |f(y)|)/2,$$

where x, y are complex variables, are $f(z) = (Az+B)^n$ and $f(z) = \exp(Az+B)$ where A, B are arbitrary complex constants and n is an arbitrary positive integer.

Now, we shall prove that (1) implies the following integral inequality:

$$(2) \qquad |1/(y-x) \int_C f(z)\,dz| \leqq (|f(x)| + |f(y)|)/2,$$

where $f = f(z)$ is an entire function of z, x, y are complex variables ($x \neq y$) and C is an arbitrary contour joining two points x and y.

To this end we shall apply the following lemma, the easy proof of which is omitted:

LEMMA 1. If $k = k(t)$ is a real-valued continuous function of a real variable t and if k is convex on $[a, b]$, then

$$1/(b-a) \int_a^b k(t)\,dt \leqq (k(a) + k(b))/2.$$

Now, we put $g(t) = |f(x+(y-x)t)|$, where $t \in [0, 1]$ and x, y are arbitrary distinct complex constants. $g(t)$ is a real-valued continuous function on $[0, 1]$. By (1) $g(t)$ is convex on $[0, 1]$. Hence, by Lemma 1 we have

$$\int_0^1 g(t)\,dt \leqq (g(0) + g(1))/2.$$

Therefore we have

$$\int_0^1 |f(x+(y-x)t)|\,dt \leqq (|f(x)| + |f(y)|)/2$$

and

$$(3) \qquad |1/(y-x) \int_L f(z)\,dz| \leqq (|f(x)| + |f(y)|)/2,$$

where L is the line segment joining the two points x and y. By Cauchy's Integral Theorem (3) implies (2).

The purpose of this note is to solve (2), i.e., to prove the following

THEOREM. *Suppose that* $f = f(z)$ *is an entire function of* z. *Then the only solutions of* (2) *are* $f(z) = (Az+B)^n$ *and* $f(z) = \exp(Az+B)$ *where* A, B *are arbitrary complex constants and* n *is an arbitrary positive integer.*

To this end we shall apply the following two lemmas:

LEMMA 2. (See [1].) Suppose that $f = f(z)$, $g = g(z)$ are entire functions of z. If $|f(z)| \leqq |g(z)|$ holds in $|z| < +\infty$, then $f(z) = Cg(z)$ where C is a complex constant with $|C| \leqq 1$.

Proof. The proof is clear from Riemann's Theorem concerning a removable singularity and Liouville's Theorem.

LEMMA 3. Suppose that $H = H(z)$ is an entire function of z. If $A(t) = |H(t\exp(i\varphi))|^2$ where t, φ are real and φ is arbitrarily fixed, then we have

$$(i) \quad A''(0) = 2Re\left(\exp(2i\varphi)\,H''(0)\,\overline{H(0)}\right) + 2\,|H'(0)|^2,$$

$$(ii) \quad A^{(4)}(0) = 2Re\left(\exp(4i\varphi)\,H^{(4)}(0)\,\overline{H(0)} + 4\exp(2i\varphi)\,H^{(3)}(0)\,\overline{H'(0)}\right) + 6\,|H''(0)|^2.$$

Proof. Since the proof is easy, we omit it.

We may now prove our theorem.

Let $F = F(z)$ be an entire function such that $F'(z) = f(z)$. By (2) we have for all complex x, y

$$2\,|F(y) - F(x)| \leqq |y - x|\,(|f(x)| + |f(y)|).$$

By a corollary of Schwarz's Inequality ($(a+b)^2 \leqq 2(a^2+b^2)$, a, b real) we have

$$2\,|F(x) - F(y)|^2 \leqq |x - y|^2\,(|f(x)|^2 + |f(y)|^2).$$

Replacing x, y by $x + y$, $x - y$, respectively, we get

$$| F(x+y) - F(x-y) |^2 \leqq 2 | y |^2 (| f(x+y) |^2 + | f(x-y) |^2) .$$

Putting $y = t \exp(i\varphi)$ where t, φ are real we have

$$| F(x+t\exp(i\varphi)) - F(x-t\exp(i\varphi)) |^2$$
$$\leqq 2t^2 (| f(x+t\exp(i\varphi)) |^2 + | f(x-t\exp(i\varphi)) |^2) .$$

Keeping x, φ arbitrarily fixed and putting

$$\begin{aligned} p(t) &= 2t^2 (| f(x+t\exp(i\varphi)) |^2 + | f(x-t\exp(i\varphi)) |^2) \\ &\quad - | F(x+t\exp(i\varphi)) - F(x-t\exp(i\varphi)) |^2 , \end{aligned}$$

$p(t)$ is a real-valued function of t and is of course four times differentiable on $|t| < +\infty$. Further $p(t)$ is an even function of t. Hence we have

(4) $$p'(0) = 0, \quad p^{(3)}(0) = 0 .$$

By Lemma 3 we have

(5) $$\begin{aligned} p''(0) &= 8 (| f(x) |^2 - | F'(x) |^2) = 0 , \\ p^{(4)}(0) &= 96 \, Re\big(\exp(2i\varphi) f''(x) \overline{f(x)}\big) + 96 | f'(x) |^2 \\ &\quad - 32 \, Re\big(\exp(2i\varphi) F^{(3)}(x) \overline{F'(x)}\big) \\ &= 64 \, Re\big(\exp(2i\varphi) f''(x) \overline{f(x)}\big) + 96 | f'(x) |^2 . \end{aligned}$$

$p(t)$ has a minimum at $t = 0$ $\big(p(t) \geqq 0$ on $|t| < +\infty$, $p(0) = 0\big)$. Hence, by (4), (5) we have $p^{(4)}(0) \geqq 0$, or

$$2 \, Re\big(\exp(2i\varphi) f''(x) \overline{f(x)}\big) + 3 | f'(x) |^2 \geqq 0 .$$

x, φ were arbitrarily fixed. An appropriate choice of φ_0 gives

$$Re\big(\exp(2i\varphi_0) f''(x) \overline{f(x)}\big) = - | f''(x) \overline{f(x)} | = - | f''(x) f(x) | .$$

Hence we have in $|x| < +\infty$

$$2 | f''(x) f(x) | \leqq 3 | f'(x) |^2 ,$$

and by Lemma 2

(6) $$f''(x) f(x) = K f'(x)^2 ,$$

where K is a complex constant with $|K| \leqq 3/2$.

Solving (6) and taking into account the fact that f is an entire function, we have

$$(7) \qquad f(z) = (Az+B)^n \quad or \quad f(z) = \exp(Az+B),$$

where A, B are complex constants and n is an arbitrary positive integer. By Theorem A, (7) satisfies (1). Since (1) implies (2) as already proved, (7) satisfies (2). Q.E.D.

REFERENCE

[1] HIROSHI HARUKI. On the functional inequality $|f((x+y)/2)| \leqq (|f(x)|+|f(y)|)/2$, *Journal of the Mathematical Society of Japan*, *16* (1964), pp. 39-41.

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