

# 5. Other proofs of Wedderburn's theorem

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number of conjugates of a subgroup is the index of its normalizer, so  $H$  has at most  $|G : H|$  conjugates in  $G$  and hence the union of the conjugates contains at most  $|G : H| (|H| - 1) + 1 = |G| - |G : H| + 1$  elements. This number is less than  $|G|$  except when  $G = H$ . Hence  $D = F$  is a field.

#### 4. FROBENIUS' THEOREM

Let  $\mathbf{R}$  denote the field of real numbers,  $\mathbf{C}$  the field of complex numbers and  $\mathbf{H}$  the division ring of quaternions. The following proof makes use of the fundamental theorem that every polynomial with coefficients in  $\mathbf{C}$  has a root in  $\mathbf{C}$ .

**THEOREM.** *Let  $D$  be a division ring which contains the real numbers  $\mathbf{R}$  in its centre and suppose that every element of  $D$  satisfies a polynomial with coefficients in  $\mathbf{R}$ . Then  $D$  is isomorphic to one of  $\mathbf{R}$ ,  $\mathbf{C}$  or  $\mathbf{H}$ .*

*Proof.* Suppose that  $D$  is not isomorphic to  $\mathbf{R}$  or  $\mathbf{C}$ . It follows that the maximal subfield  $F$  of  $D$  is isomorphic to  $\mathbf{C}$ , the centre  $K$  of  $D$  is isomorphic to  $\mathbf{R}$  and  $F = K(i)$  where  $i^2 = -1$ . Let  $j$  be an eigenvector of  $T_i$  corresponding to the eigenvalue  $-i$ . Then  $ji = -ij$  and  $j^2$  commutes with  $j$  and  $F$ . From (2.2) and (2.3) the elements 1 and  $j$  form an  $F$ -basis for  $D$  and therefore  $j^2 = \alpha$  belongs to  $K$ . If  $\alpha = \beta^2$  for some  $\beta \in K$  then  $(j - \beta)(j + \beta) = 0$  and  $j$  belongs to  $K$ , which is not the case; hence  $\alpha = -\beta^2$  for some  $\beta \in K$ . Replacing  $j$  by  $j\beta^{-1}$  we obtain a  $K$ -basis 1,  $i$ ,  $j$ ,  $ij$  for  $D$  such that  $i^2 = j^2 = -1$  and  $ij = -ji$ . That is,  $D$  is isomorphic to  $\mathbf{H}$ .

An almost identical argument shows that if the dimension of  $D$  over its centre  $K$  is 4 and the characteristic is not 2, then  $D$  has a  $K$ -basis 1,  $i$ ,  $j$ ,  $ij$  where  $i^2 = \alpha$ ,  $j^2 = \beta$  and  $ij = -ji$  for some  $\alpha, \beta \in K$ .

#### 5. OTHER PROOFS OF WEDDERBURN'S THEOREM

The original proofs of the theorem of §3 were given first by Wedderburn [15] in 1905 and then by Dickson [5] in the same year; they depend on certain divisibility properties of the integers. The neatest proof along these lines is that of Witt [16]. Elementary proofs which avoid the use of such number theory have been given by Artin [1] and Herstein [7]. And

proofs which deduce the theorem using finite group theory have been given by Zassenhaus [17], Brandis [3] and Scott [11, p. 426].

Perhaps the most interesting proofs are those which present the result as a consequence of a more general theory. There are two such proofs in the book of van der Waerden [14]: the first (on p. 203) uses the theory of central simple algebras, the second (sketched on p. 215) relates the theorem to cohomology and the Brauer group (see also, Serre [12, p. 170]). The theorem is also a consequence of the work of Tsen [13] and Chevalley [4]. Further comments on the history of the theorem can be found in an article by Artin [2] and in the book by Herstein [8] where many interesting generalisations are also given. One such generalization is a theorem of Jacobson: a division ring in which  $x^{n(x)} = x$  for all  $x$  is commutative. Laffey [10] has recently given an elementary proof of this using Wedderburn's theorem and linear algebra similar to that used here. See also [18].

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