

5. Conclusion

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **20 (1974)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **30.06.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

The discriminant is $4p^2(p - \alpha^2 r^2)$ which cannot vanish, so that, as before, the first factor in (10) must be zero, and we have

$$(11) \quad (\alpha^2 + \beta^2)r^2 - p = 0$$

which is a contradiction since $\alpha^2 + \beta^2 > 1$ and we are supposing that $|r| > 1$.

Therefore we cannot have $|r| > 1$, $K \neq 0$, and $L \neq 0$. If $|r| = 1$ we see that $K = L = 0$ since $|x - \alpha r| < p$ and $|y - \beta r| < p$ in this case. If $|r| > 1$ with $K = L = 0$ we would have $x = \alpha r$, $y = \beta r$ and hence $(x, y) > 1$, whereas x and y are relatively prime. Finally it remains to consider the possibility of having $|r| > 1$ with one of K and L zero, the other nonzero. This if we suppose that $|r| > 1$, $K = 0$, $L \neq 0$, we obtain (9) which, as we have seen, leads to a contradiction. On the other hand the supposition that $|r| > 1$ with $K \neq 0$, $L = 0$ implies that (11) would hold with $r^2 > 1$.

We conclude that $|r| = 1$, $K = 0$ and $L = 0$. Hence $x = \pm \alpha$, $y = \pm \beta$ and $\alpha^2 + \beta^2 = p$.

In [1], Corollary 2, we observed that if $p = x^2 + y^2$ then, in our notation, y is a quadratic residue of p . Collecting our results we have the

COROLLARY. Let $p = x^2 + y^2$ where p is a prime of the form $4n + 1$ with x and y given by (3) and (4). Then $\left(\frac{x}{p}\right) = \left(\frac{2}{p}\right)$ and $\left(\frac{y}{p}\right) = 1$.

5. CONCLUSION

We saw that $x = \pm \alpha$, $y = \pm \beta$. When $p = 13$ we have $y = -3$, $\beta = -3$; when $p = 29$, $y = -5$, $\beta = 5$, and when $p = 41$, $y = 5$, $\beta = 5$. Hence the sign of y , determined by the approximants to a continued fraction depends on the integer m , the number of terms in the finite segment of (2) which is used, can agree with that of β or be opposite that of β . The same applies to x and α . In [1], Theorem 1, we gave a construction which always gives positive values for x and y . Other various constructions, as we have seen, do not have this property.

Finally we comment on the numbers $\frac{(2n)!}{2(n!)^2}$ which we denote by a_n for $n = 1, 2, 3, \dots$

The members of the sequence $\{a_n\}$ are related to the numbers $b_{n+1} = \frac{(2n)!}{(n+1)!n!}$, $n = 0, 1, 2, \dots$, which, as mentioned by Becker [2], have a variety of applications. Birkhoff [3] pointed out that b_n is an integer for every positive integer n , and noted the recurrence relation $b_n = \sum_{i=1}^{n-1} b_i b_{n-i}$; a relation which was also obtained by Wedderburn [10].

The results of this note depend on the fact that a_n is an integer, at least when $p = 4n + 1$ is a prime. Although it is known that a_n is an integer for every positive integer n , we can see that this also follows readily from [3]. For we have $2a_n = (n+1)b_{n+1}$. If n is even, it follows that b_{n+1} is even since $(2, n+1) = 1$. Therefore $a_n = (n+1) \frac{b_{n+1}}{2}$ is an integer. If n is odd then $2 \mid (n+1)$ and in this case also $a_n = \frac{n+1}{2} b_{n+1}$ is an integer. A list of values for a_n can be obtained from the second column of a table in [2], page 699, headed N_n , by multiplying the $(n+1)$ st member by $\frac{n+1}{2}$.

REFERENCES

- [1] BARNES, C. W. The Representation of Primes of the Form $4n + 1$ as the Sum of Two Squares, *L'Enseignement Mathématique*, 18 (1972), pp. 289-299.
- [2] BECKER, H. W. Discussion of Problem 4277, *American Mathematical Monthly*, 56 (1949), pp. 697-699.
- [3] BIRKHOFF, Garrett. Problem 3674, *American Mathematical Monthly*, 42 (1935), pp. 518-521.
- [4] CAUCHY, A. *Sur les Formes quadratiques de certaines Puissances des Nombres Premiers ou du quadruple de ces Puissances*, Œuvres complètes 1^{re} série, tome 3, pp. 390-437.
- [5] DAVENPORT, H. *The Higher Arithmetic*, Hutchinson's University Library, London, 1952.
- [6] GAUSS, C. F. Werke Bd. 2, S. 90-91.
- [7] JACOBSTHAL, Ernst. Über die Darstellung der Primzahlen der Form $4n + 1$ als Summe Zweier Quadrate, *Journal für die Reine und Angewandte Mathematik*, Band 132, (1907), pp. 238-245.
- [8] LEGENDRE, A. M. *Théorie des Nombres*. Troisième édition, Paris, 1830.
- [9] PERRON, Oskar. *Die Lehre von den Kettenbrüchen*. Chelsea, New York, 1951.
- [10] WEDDERBURN, J. H. M. The Functional Equation $g(x^2) = 2\alpha x + [g(x)]^2$, *Annals of Mathematics*, (2), 24, (1922-1923), pp. 121-140.

(Reçu le 1^{er} mai 1973)

C. W. Barnes
 Department of Mathematics
 University of Mississippi
 Mississippi, 38677, USA