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1. Ordered Fields

In this section we shall review some well known results about ordered fields and state (and prove) some not so well known ones.

1. Every ordered field contains the rational number system Q as a subordered field.
2. The real number system R constitutes a complete (in the upper bound sense) ordered field.
3. Any two complete ordered fields are isomorphic with respect to $+$, $\times$, and $<$.
4. Any complete ordered field R' is Archimedean; that is, to each $a \in R'$ there exists a natural number n such that $n > a$.
5. There exists ordered fields which contain the real number system as a proper subordered field.
6. Any ordered field F which contains the reals as a proper subordered field must be non-Archimedean and, consequently, cannot be complete.

Proofs:

The first four are to be found in most advanced calculus books.

In 5 the existence of the desired field can be shown by considering $R(X)$, the field of rational functions in one indeterminate with real coefficients. R can be identified with the polynomials of degree 0. To define an ordering on $R(X)$ it is sufficient to specify the positive members, then we can define the ordering $<$ by the rule: $\alpha < \beta$ iff $\alpha - \beta$ is positive. Take for the positive elements those rational functions which can be represented as a quotient of two polynomials both of which have positive leading coefficient. This particular ordered field will play no role, however, in our subsequent discussions.

We prove 6 by contradiction. Suppose F is Archimedean. Choose α such that $\alpha \in F$ and $\alpha \notin R$. Since F is Archimedean there exists a natural number n such that $|\alpha| < n$. (Recall that the notion of absolute value is meaningful in any ordered field.) Let $A = \{x \in R \mid x \leqq |\alpha|\}$. A is bounded above by n, so A has a smallest real upper bound s. Now since s is real and α isn't, we have that $s \neq |\alpha|$ and we can form the reciprocal of $s - |\alpha|$. Since F is assumed to be Archimedean, there exists a natural number k such that

$$k > \frac{1}{|s - |\alpha||}$$

and from this we get

$$s - |\alpha| > \frac{1}{k} \quad or \quad |\alpha| - s > \frac{1}{k}.$$

Case 1. $s - |\alpha| > \frac{1}{k}$. In this case $s - \frac{1}{k} > |\alpha|$, but then by definition of A we see that $s - \frac{1}{k}$ is a real upper bound of A. Moreover, $s - \frac{1}{k}$ is smaller than the *least* upper bound s, which is absurd.

Case 2. $|\alpha| - s > \frac{1}{k}$. Then $|\alpha| > s + \frac{1}{k}$, so $s + \frac{1}{k} \in A$ by definition of A. But s is an upper bound for A so $s + \frac{1}{k} \leqq s$ from which follows $k \leqq 0$; but this contradicts the fact that k is a natural number.

(Q.E.D.)

2. Ordered Fields Which Properly Contain the Reals

In this section we shall assume that F is an ordered field which has the real numbers R as a proper subordered field. We have already seen that F must be non-Archimedean. N will be used to denote the set of natural numbers.

An element $a \in F$ is said to be

infinitesimal if $|a| < r$ for each positive real r.

finite if $|a| \leqq r$ for some real r.

infinite if $|a| > r$ for every real r.

The number 0 is certainly infinitesimal, but it is easy to see that there are also non-zero infinitesimals and infinites as follows:

F being non-Archimedean must contain an element b such that $n \leqq b$ for all $n \in N$. This implies that $n < b$ all $n \in N$ and, in fact, $r < b$ all $r \in R$. Thus b is infinite and $\frac{1}{b}$ is infinitesimal.