

# 3. The Main Theorem

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### 3. THE MAIN THEOREM

All that has been said here so far was known long before the advent of Non-Standard Analysis. Now we come to the heart of the matter—the key theorem. It was first obtained by Robinson as a corollary to the so-called Compactness Theorem of mathematical logic. Later proofs were given by means of the ultraproduct construction which also has its roots in mathematical logic. We shall content ourselves with a mere statement of the result.  $R$  as usual denotes the real number system and  $N$  the natural number system.

**THEOREM 3.1 (MAIN THEOREM).** There is a set  $R^*$  for which all of the following hold:

1.  $R$  is a proper subset of  $R^*$ .
2. To each  $n$ -place function  $f(x_1, \dots, x_n)$  from  $R^n$  to  $R$  ( $n \geq 1$ ), there corresponds a certain function  $f^*(x_1, \dots, x_n)$  from  $(R^*)^n$  to  $R^*$  which agrees with  $f(x_1, \dots, x_n)$  on  $R^n$ .
3. To each  $n$ -place relation  $A(x_1, \dots, x_n)$  on  $R$  ( $n \geq 1$ ), there corresponds a certain relation  $A^*(x_1, \dots, x_n)$  on  $R^*$  which agrees with  $A(x_1, \dots, x_n)$  on  $R$ . The relation corresponding to the equality relation on  $R$  is the equality relation on  $R^*$ .
4. Every statement  $\mathcal{S}$  formulated in terms of
  - i) particular (fixed) real numbers
  - ii) particular (fixed) real functions
  - iii) particular (fixed) real relations
  - iv) variables ranging over  $R$
  - v) logical operations and quantifiers

is true about  $R$  if and only if the statement  $\mathcal{S}^*$  obtained from it by

- a) replacing each  $f(x_1, \dots, x_n)$  by  $f^*(x_1, \dots, x_n)$
- b) replacing each  $A(x_1, \dots, x_n)$  by  $A^*(x_1, \dots, x_n)$
- c) letting the variables range over  $R^*$

is true about  $R^*$ .

It turns out that there are many such  $R^*$ . From here on out it will be assumed that we are fixing on one of them.

The theorem is quite a mouthful and it must be admitted that our formulation of it suffers from a little imprecision owing to the fact that we

never said what a statement is <sup>1)</sup>. A few examples, however, should nail the idea down. Let us add for emphasis that we are only allowing statements of finite length.

Example 3.1. Consider the statement

$$(\forall x) (0 + x = x)$$

which is true when the variables range over  $R$ ; it asserts that the particular real number 0 is a left identity for the  $+$  operation. By the Main Theorem, the statement

$$(\forall x) (0 +^* x = x)$$

must be true when the variables range over  $R^*$ ; thus 0 is also a left identity for the  $+^*$  operation on  $R^*$ .

Example 3.2. Let  $f$  be a particular function from  $R$  to  $R$  which is an “onto” function. Then the statement

$$(\forall y) (\exists x) (f(x) = y)$$

is true when the variables range over  $R$ . Therefore by the Main Theorem the statement

$$(\forall y) (\exists x) (f^*(x) = y)$$

is true when the variables range over  $R^*$ ; that is, the function  $f^*$  is onto  $R^*$ .

Henceforth instead of saying “true when the variables range over  $R$ ”, we shall simply say “true in  $R$ ”.

In subsequent discussions members of  $R$  will be called *standard* numbers, while members of  $R^* - R$  will be called *non-standard* numbers. Likewise functions from  $R^n$  to  $R$  ( $n \geq 1$ ), relations on  $R$ , and subsets of  $R$  will be called *standard* functions, relations and subsets. Some writers refer to members of  $R^*$  as real numbers, but we shall reserve the term for members of  $R$ . Thus standard number and real number have the same meaning here.

Statements which can be formulated in the manner prescribed in the hypothesis of the Main Theorem are called *admissible* statements. You should convince yourself, by writing them out if necessary, that all the axioms of an ordered field are admissible; moreover, they are true about  $R$  (because

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<sup>1)</sup> Using the terminology of formal logic the class of statements in question can be defined as the class of closed well-formed formulae of a generalized first-order language having distinct individual, function, and relation constants corresponding to each real, real function and real relation.

$R$  is an ordered field). Now by the Main Theorem they are all true about  $R^*$  if we put the stars on the symbols  $+$ ,  $\times$ ,  $<$ . But this is just a way of saying that  $R^*$  is an ordered field with respect to  $+^*$ ,  $\times^*$ ,  $<^*$ . Moreover since the theorem provided that these agree with  $+$ ,  $\times$ ,  $<$  respectively on  $R$ , we can say that  $R^*$  is an ordered field which has  $R$  as a proper subordered field. Now recalling results from our review on ordered fields we have that  $R^*$  is non-Archimedean and is not complete.

Now at this point you might be getting a bit suspicious. You might ask: “Why not show the completeness of  $R^*$  (and thus get a paradox) by taking the assertion that  $R$  is complete, and then use the Main Theorem to conclude that  $R^*$  is complete?” The catch is that the Completeness Axiom has a logical structure fundamentally different from the ordered field axioms. It’s not an admissible statement! Its form is

$$(\forall S) (S \text{ bounded} \rightarrow \dots\dots\dots)$$

that is, it has a variable ranging over the family of *subsets* of  $R$ . Recall, the variables in an admissible statement must range over  $R$ .

With respect to the Archimedean property the catch is a little different. Using the symbols  $N(y)$  to denote the particular one-place relation—“ $y$  is a natural number,” we *can* assert that  $R$  is Archimedean by the admissible statement

$$(\forall x) (\exists y) (N(y) \wedge x < y);$$

thus

$$(\forall x) (\exists y) (N^*(y) \wedge x <^* y)$$

is true in  $R^*$ , but it doesn’t necessarily say that  $R^*$  is Archimedean. The  $y$  which is asserted to exist, and for which  $N^*(y)$  holds, might be in  $R^* - R$ ; that is, it might be non-standard. To be sure, it does say that  $R^*$  has some sort of formal Archimedean-like property, but if in the definition of Archimedean one requires that  $y$  actually be a member of  $N$  (and we shall), then  $R^*$  isn’t Archimedean.

In the sequel it may at times be too repetitious to write statements first without the stars  $*$ , and then with them. It will usually be clear from the context whether the stars are intended. Thus if we were to say that

$$(\forall x) (\forall y) (x < y \rightarrow f(x) < f(y))$$

is true in  $R^*$ , then you are to understand that we are really talking about

$<^*, f^*$  and the variables are to range over  $R^*$ . Sometimes we shall put on some of the stars for emphasis.

#### 4. FIXED SUBSETS

Let  $S$  be a particular (fixed) subset of  $R$ . We can identify  $S$  with the one-place relation  $S(x)$  which holds for a given  $x$  if and only if  $x \in S$ ; that is,

$$S = \{ x \in R \mid S(x) \}.$$

We can now define a set  $S^* \subseteq R^*$  by

$$S^* = \{ x \in R^* \mid S^*(x) \}.$$

Clearly  $S \subseteq S^*$  because  $S^*(x)$  agrees with  $S(x)$  on  $R$ . We shall often write

$$x \in S \text{ instead of } S(x)$$

and

$$x \in S^* \text{ instead of } S^*(x).$$

The upshot of the above is that the Main Theorem also provides for an extension  $S^*$  for each  $S \subseteq R$  and that we can allow as admissible statements those which involve the sentence fragment  $x \in S$ ; in “lifting” statements from  $R$  to  $R^*$  we replace the fragment  $x \in S$  by  $x \in S^*$ . Warning! The requirement that admissible statements be permitted only variables ranging over  $R$  hasn’t been altered. In a given statement the functions, relations, and subsets must remain fixed!

Example 4.1. Let  $S = \{ x \in R \mid x < 6 \}$ . Now

$$(\forall x) (x \in S \leftrightarrow x < 6) \text{ is true in } R$$

so

$$(\forall x) (x \in S^* \leftrightarrow x <^* 6) \text{ is true in } R^*.$$

Thus

$$S^* = \{ x \in R^* \mid x <^* 6 \}.$$

Furthermore  $S^*$  is a proper extension of  $S$ , because for any infinitesimal  $\varepsilon$ , the number  $5 + \varepsilon$  is a member of  $S^*$ , but not being a standard number, it can’t be a member of  $S$ .