4. Fixed Subsets

$<^*, f^*$ and the variables are to range over R^*. Sometimes we shall put on some of the stars for emphasis.

4. Fixed Subsets

Let S be a particular (fixed) subset of R. We can identify S with the one-place relation $S(x)$ which holds for a given x if and only if $x \in S$; that is,

$$S = \{ x \in R \mid S(x) \}.$$

We can now define a set $S^* \subseteq R^*$ by

$$S^* = \{ x \in R^* \mid S^*(x) \}.$$

Clearly $S \subseteq S^*$ because $S^*(x)$ agrees with $S(x)$ on R. We shall often write

$$x \in S \text{ instead of } S(x)$$

and

$$x \in S^* \text{ instead of } S^*(x).$$

The upshot of the above is that the Main Theorem also provides for an extension S^* for each $S \subseteq R$ and that we can allow as admissible statements those which involve the sentence fragment $x \in S$; in "lifting" statements from R to R^* we replace the fragment $x \in S$ by $x \in S^*$. Warning! The requirement that admissible statements be permitted only variables ranging over R hasn't been altered. In a given statement the functions, relations, and subsets must remain fixed!

Example 4.1. Let $S = \{ x \in R \mid x < 6 \}$. Now

$$(\forall x)\,(x \in S \leftrightarrow x < 6) \text{ is true in } R$$

so

$$(\forall x)\,(x \in S^* \leftrightarrow x <^* 6) \text{ is true in } R^*.$$

Thus

$$S^* = \{ x \in R^* \mid x <^* 6 \}.$$

Furthermore S^* is a proper extension of S, because for any infinitesimal ε, the number $5 + \varepsilon$ is a member of S^*, but not being a standard number, it can't be a member of S.

For any finite set $T \subseteq R$ we can show $T = T^*$. Suppose $T = \{a_1, ..., a_n\}$ then

$$(\forall x)\ (x \in T \leftrightarrow [x = a_1 \vee x = a_2 \vee \ldots \vee x = a_n])$$

is true in R, thus

$$(\forall x)\ (x \varepsilon T^* \leftrightarrow [x = a_1 \vee x = a_2 \vee \ldots \vee x = a_n])$$

is true in R^*; that is, $T^* = \{a_1, ..., a_n\}$.

Although the Main Theorem makes no mention of functions f whose domain is a proper subset $D \subset R$. We can define a function $f^* : D^* \to R^*$ in a natural way. Arbitrarily extend f to a function g which is defined on all of R; then let f^* be the restriction of g^* to D^*. This definition is easily seen to be independent of the way f is extended.

5. Infinite Natural Numbers

We have seen in the last section that each particular $S \subseteq R$ has associated with it a certain extension $S^* \subseteq R^*$. We now consider the case when we take S to be N, the set of natural numbers. One can see that N^* actually has some non-standard members as follows. The statement "N is unbounded" is true in R and can be formulated as the admissible statement

$$(\forall x)\ (\exists y)\ (y \in N \wedge y > x);$$

therefore

$$(\forall x)\ (\exists y)\ (y \in N^* \wedge y > x)$$

is true in R^*. It asserts that N^* is an unbounded subset of R^*. If we let α be an infinite member of R^*, then N^* must have an even larger member which, of course, is also infinite and non-standard.

We can show that all the non-standard members of N^* are infinite in the following way. Formulate as admissible statements each of the infinitely many assertions:

"All natural numbers are greater than 0."
"No natural numbers lie between 0 and 1."
"No natural numbers lie between 1 and 2."
etc.