6. LIMITS, CONTINUITY, BOUNDEDNESS, AND COMPACTNESS

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Each of those statements then must be true in R^* when we read N^* instead of N, so each member of $N^* - N$ must be greater than all the real numbers.

In view of the above we call the non-standard members of N^* *infinite natural numbers.*

Now it is easy to show that each infinite natural number has an immediate successor in N^* (because of the corresponding result for N), and each infinite natural number has an infinite immediate predecessor in N^*. N^* isn't well ordered because if α is an infinite natural number, the chain

$$\alpha > \alpha - 1 > \alpha - 2 > \cdots$$

has no least member. Here again one might be tempted to use the Main Theorem to infer that N^* is well ordered because N is; however, the statement that N is well ordered is not admissible by virtue of its having a variable ranging over subsets. It reads:

"Every non-empty subset of N..."

Concepts such as even number, odd number, and prime number are all meaningful for infinite natural numbers; indeed, if $E \subseteq N$ is the set of even numbers, then E^* is the set of even numbers of N^*.

It will be shown later that N^* is uncountably infinite.

6. Limits, Continuity, Boundedness, and Compactness

Now we show that R^* provides the appropriate machinery for formulating concepts from the Calculus in an intuitive and direct way. Consider, for example, the limit concept. The $\varepsilon - \delta$ definition of $\lim\limits_{x \to c} f(x) = L$ seems to be a roundabout way of saying that for x infinitely close to but not equal to c, $f(x)$ will be infinitely close to L. Now it makes sense to say it just that way provided we are talking about $f^*(x)$. It not only makes sense, but as the next theorem shows, saying it that way actually gives a correct characterization of $\lim\limits_{x \to c} f(x) = L$.

Theorem 6.1. Let f be a standard function defined on a standard open interval (a, b) having c as an interior point. Suppose further that L is standard, then

(a) $\lim\limits_{x \to c} f(x) = L$ if and only if $c \neq x \approx c$ implies $f^*(x) \approx L$.

(b) $f(x)$ is continuous at c if and only if $x \approx c$ implies $f^*(x) \approx f(c)$.

PROOF. At this point there should be no confusion if we sometimes omit the symbol *. Part (b) follows immediately from part (a).

Part (a), direction $\Rightarrow$. Suppose $\lim_{x \to c} f(x) = L$. To show that $c \neq x \approx c$ implies $f(x) \approx L$. Let x_0 be given such that $c \neq x_0 \approx c$, to show $f(x_0) \approx L$ we must show that $f(x_0) - L$ is infinitesimal; that is, we must show that $|f(x_0) - L| < \varepsilon$ for each positive real ε. Let arbitrary but fixed positive real ε_0 be given. Must show that the statement

$$(1) \qquad |f(x_0) - L| < \varepsilon_0$$

is true in R^*. By definition of limit we know that there exists a positive real δ such that $0 < |x - c| < \delta$ implies that $|f(x) - L| < \varepsilon_0$. Let δ_0 be such a δ, then the statement

$$(\forall x)\,(0 < |x - c| < \delta_0 \rightarrow |f(x) - L| < \varepsilon_0)$$

is true in R; therefore, it's true in R^*. In particular then the statement

$$(2) \qquad 0 < |x_0 - c| < \delta_0 \rightarrow |f(x_0) - L| < \varepsilon_0$$

is true in R^*. Now from $c \neq x_0 \approx c$ we know that $0 < |x_0 - c| < r$ for each positive real r, so in particular $0 < |x_0 - c| < \delta_0$ is true in R^*. This with (2) gives $|f(x_0) - L| < \varepsilon_0$ which is the statement (1) we needed to show.

Part (a), direction $\Leftarrow$. The argument is rather novel. Assume that

$$(3) \qquad c \neq x \approx c \text{ implies } f(x) \approx L.$$

Let arbitrary but fixed positive real ε_0 be given, must show that the statement

$$(\exists \delta)\,(\delta > 0 \wedge (\forall x)\,[0 < |x - c| < \delta \rightarrow |f(x) - L| < \varepsilon_0])$$

is true in R. Now this is an admissible statement, so it suffices to show that it is true in R^*. As a statement about R it is an assertion that there exists a real δ with certain properties. In showing it to be true in R^* we are permitted to seek the δ from among the positive infinitesimals if we so desire. We now show that any positive infinitesimal δ will do. Let δ_0 be a positive infinitesimal. Must show that the statement

$$(\forall x)\,[0 < |x - c| < \delta_0 \rightarrow |f(x) - L| < \varepsilon_0]$$

is true in R^*. Let arbitrary $x_0 \in R^*$ be given, must show that

(4) $$0 < |x_0 - c| < \delta_0 \rightarrow |f(x_0) - L| < \varepsilon_0$$

is true. Assume that the left side of the arrow is true; that is, assume

(5) $$0 < |x_0 - c| < \delta_0;$$

we want to show under this assumption that the right side is true. Since δ_0 is infinitesimal we can infer from (5) that $c \neq x_0 \approx c$, but then by (3) $f(x_0) \approx L$, that is, $f(x_0) - L$ is infinitesimal. Since ε_0 is positive real, $|f(x_0) - L| < \varepsilon_0$, which is the right side of the arrow in (4)

(Q.E.D.)

Example 6.1. Suppose we want to show that the composition of two continuous functions is continuous. The standard proof is easy enough, but the following non-standard proof is more direct and intuitive. Let $g(x)$ be continuous at c and $f(x)$ continuous at $g(c)$. Let $x \approx c$ be given. Since g is continuous at c, $g(x) \approx g(c)$. Since f is continuous at $g(c)$, $f(g(x)) \approx f(g(c))$.

For functions whose domain is not an interval but some set S, appropriate modifications of the argument in the preceding theorem gives the theorem below.

THEOREM 6.2. The standard function $f(x)$ with standard domain S is continuous at the standard point c if and only if whenever x is a point of S^* infinitely close to c, $f^*(x) \approx f(c)$.

The notion of a function being bounded has a very useful non-standard characterization. By bounded we mean, as usual, that there is a standard bound.

THEOREM 6.3. A standard function f is bounded on a standard set S if and only if $f^*(x)$ is finite for each $x \in S^*$.

PROOF. Direction $\Rightarrow$. Suppose f is bounded on S. Then there exists a standard number r_0 such that the sentence

$$(\forall x)(x \in S \rightarrow |f(x)| \leq r_0)$$

is true in R and therefore also in R^*. Thus if $x_0 \in S^*$ we have that $|f^*(x_0)| \leq r_0$. By definition of finite this means $f^*(x_0)$ is finite.

Direction $\Leftarrow$. Suppose $f^*(x)$ is finite all $x \in S^*$. We want to show that the statement

(6) $$(\exists t)(\forall x)(x \in S \rightarrow |f(x)| \leq t)$$

is true in R. If suffices to show that it is true in R^*, but in R^* we can take the t to be any positive infinite number. Then since each $f^*(x)$ is finite we have that $|f^*(x)| \leqq t$ all $x \in S^*$, that is (6) is true in R^*.

(Q.E.D.)

Note that if $[c, d]$ is a standard closed interval, then $[c, d]^*$ is the closed interval $\{x \in R^* \mid c \leqq x \leqq d\}$; this is because the statement

$$(\forall x)\,(x \in [c, d] \leftrightarrow (c \leqq x \wedge x \leqq d))$$

is true in R and, therefore, in R^*. A similar result holds for the other types of intervals.

Compare the following non-standard proof of a well known theorem with the standard proofs you know!

THEOREM 6.4. If the standard function f is continuous at each point of the standard closed interval $[c, d]$, then f is bounded there.

PROOF. In view of the preceding theorem we have only to show that $f^*(x)$ is finite for all $x \in [c, d]^*$. Let $x \in [c, d]^*$ be given. Clearly x is finite and according to Theorem 2.1 it is infinitely close to a standard point x_0. It is easy to see that $x_0 \in [c, d]$. By continuity $f^*(x) \approx f(x_0)$. Since f is a standard function, $f(x_0)$ is finite, but then $f^*(x)$ being infinitely close, is also finite.

(Q.E.D.)

As is well known, the above theorem fails for open intervals. An attempted proof would break down when we try to assert that $x_0 \in (c, d)$. It might just happen that x_0 is one of the end points.

Note that in the above proof, the only property of the closed interval used there is:

"Every point of $[c, d]^*$ is infinitely closed to some point of $[c, d]$." Thus the theorem can be generalized by replacing $[c, d]$ with a set S having the same property, namely

(7) "Every point of S^* is infinitely close to some point of S."

We show further that this property of S is a necessary condition on S for *all* continuous functions on S to be bounded. By contrapositive assume (7) fails; under this assumption we will produce a continuous function on S which takes on an infinite value at a point in S^* from which it would follow that the function isn't bounded on S. To say that (7) fails would mean that there exists $x_0 \in S^*$ such that for all $y \in S$, $x_0 \not\approx y$. If x_0 is infinite, then

$f(x) = x$ is a continuous function which is infinite at $x = x_0$. On the other hand, if x_0 were finite then x_0 is infinitely close to some standard number y_0, and since (7) fails $y_0 \notin S$. Thus the function $f(x) = \frac{1}{x - y_0}$ is continuous on S (the denominator can't be zero for $x \in S$ because $y_0 \notin S$); moreover, $f(x_0)$ is infinite since $x_0 - y_0$ is a non-zero infinitesimal.

It is known in the study of the topology of the real line that a necessary and sufficient condition for a set S to have the property that all continuous functions on it be bounded is that S be compact. We've just shown that (7) is also necessary and sufficient, so this establishes the following theorems.

THEOREM 6.5. A set $S \subseteq R$ is compact if and only if every point of S^* is infinitely close to a point of S.

THEOREM 6.6 If the standard function $f(x)$ is continuous on a standard compact set S, then $f(x)$ is bounded there.

It turns out that in applying the methods of Non-standard Analysis to the subject of General Topology, the characterization of compactness given by Theorem 6.5 still holds.

The theorem below gives a very nice characterization of the notion of a uniformly continuous function. We shall not deny you the pleasure of trying to prove it yourself. The proof of Theorem 6.1 should provide the inspiration.

THEOREM 6.7. A standard function is uniformly continuous on the standard set S if and only if $x \approx y$ implies $f^*(x) \approx f^*(y)$ for all $x, y \in S^*$.

Using the above theorem we can quickly dispatch the following.

THEOREM 6.8. A standard function f continuous on a compact standard set S is uniformly continuous on S.

PROOF. Let $x, y \in S^*$ be given such that $x \approx y$. By compactness of S there exists $x_0 \in S$ such that $x \approx x_0$. Since $\approx$ is an equivalence relation $x \approx x_0 \approx y$. Now by continuity $f^*(x) \approx f(x_0) \approx f^*(y)$, therefore $f^*(x) \approx f^*(y)$.

7. INFINITE SEQUENCES

An infinite sequence $\{a_n\}$ can be thought of as a function from N into R. Accordingly the Main Theorem provides for an extension function from N^* into R^*. Put differently, after we exhaust all the terms with finite