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$$f(c) \approx f(a_i^\alpha) \text{ and } f(c) \approx f(a_{i-1}^\alpha).$$

Taking this together with the fact (seen already) that

$$f(a_i^\alpha) \geqq 0 \text{ and } f(a_i^\alpha) < 0$$

we have (in summary) that $f(c)$ is a standard number infinitely close to a negative number and a non-negative number. Thus $f(c) = 0$.

(Q.E.D.)

9. Derivatives

Let $f(x)$ be a standard function defined on a standard open interval (a, b) and having the point x_0 as an interior point. Using the non-standard characterization of limit, the condition that $f(x)$ be differentiable at x_0 is that there exist a standard number L such that

$$\frac{f(x_0+dx) - f(x_0)}{dx} \approx L$$

for all non-zero infinitesimals dx. L, of course, will be the derivative. If $f(x)$ is differentiable, then writing $dy = f(x_0+dx) - f(x_0)$ we have (using the notation for "standard part" introduced in Section 2) $°(\frac{dy}{dx}) = f'(x_0)$. This says that the quotient of the infinitesimal increments need not in general be the derivative, but it must be infinitely close to it.

Example 9.1. Suppose we wish to calculate the derivative of $f(x) = x^2$. Let dx be an arbitrary non-zero infinitesimal, then

$$\frac{dy}{dx} = \frac{(x+dx)^2 - x^2}{dx}$$

After squaring and cancelling we get, $\frac{dy}{dx} = 2x + dx \approx 2x$ therefore

$$°(\frac{dy}{dx}) = 2x.$$

That is, the function x^2 is differentiable with derivative $2x$.

Example 9.2. Let's see how to prove the Chain Rule! Suppose $f(x)$ and $g(x)$ are differentiable at the appropriate places and we wish to show that the function $h(x) = f(g(x))$ is differentiable with derivative $h'(x) = f'(g(x))\,g'(x)$. For any non-zero infinitesimal dx, write $dg = g(x+dx) - g(x)$ and $dh = h(x+dx) - h(x)$ then

$$dh = f(g(x + dx)) - f(g(x)) = f(g(x) + dg) - f(g(x)).$$

We want to show that for any non-zero infinitesimal dx,

$$\frac{dh}{dx} \approx f'(g(x))\,g'(x)\,. \tag{1}$$

Let non-zero infinitesimal dx be given. By continuity of $g(x)$, dg is also infinitesimal.

Case 1. $dg = 0$. Then $dh = 0$, so ${}^{\circ}\!\left(\frac{dg}{dx}\right) = g'(x) = 0$ and $\frac{dh}{dx} = 0$. Thus both sides of (1) are zero, so (1) holds.

Case 2. $dg \neq 0$. Then $\frac{dh}{dx} = \frac{dh}{dg} \cdot \frac{dg}{dx}$ that is

$$\frac{dh}{dx} = \frac{f(g(x)+dg) - f(g(x))}{dg} \cdot \frac{g(x+dx) - g(x)}{dx}\,. \tag{2}$$

The two factors of the right side of (2) are infinitely close to $f'(g(x))$ and $g'(x)$ respectively. Now using the rules given in Section 2 for manipulating the symbol $\approx$ we get

$$\frac{dh}{dx} \approx f'(g(x)) \cdot g'(x)$$

as desired.

10. Integration

Let $f(x)$ be a standard function integrable on the standard interval $[a, b]$. For each standard n let

$$a = a_0^n < a_1^n < \cdots < a_n^n = b$$

be a partition of the interval into n subintervals of equal length. The Riemann sums

$$S_n = \sum_{i=1}^{n} f(a_i^n)\,(a_i^n - a_{i-1}^n)$$