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$$f(c) \approx f(a_i^\alpha) \text{ and } f(c) \approx f(a_{i-1}^\alpha).$$

Taking this together with the fact (seen already) that

$$f(a_i^\alpha) \geq 0 \text{ and } f(a_i^\alpha) < 0$$

we have (in summary) that  $f(c)$  is a standard number infinitely close to a negative number and a non-negative number. Thus  $f(c) = 0$ .

(Q.E.D.)

## 9. DERIVATIVES

Let  $f(x)$  be a standard function defined on a standard open interval  $(a, b)$  and having the point  $x_0$  as an interior point. Using the non-standard characterization of limit, the condition that  $f(x)$  be differentiable at  $x_0$  is that there exist a standard number  $L$  such that

$$\frac{f(x_0 + dx) - f(x_0)}{dx} \approx L$$

for all non-zero infinitesimals  $dx$ .  $L$ , of course, will be the derivative. If  $f(x)$  is differentiable, then writing  $dy = f(x_0 + dx) - f(x_0)$  we have

(using the notation for “standard part” introduced in Section 2)  $\circ\left(\frac{dy}{dx}\right)$

$= f'(x_0)$ . This says that the quotient of the infinitesimal increments need not in general be the derivative, but it must be infinitely close to it.

Example 9.1. Suppose we wish to calculate the derivative of  $f(x) = x^2$ . Let  $dx$  be an arbitrary non-zero infinitesimal, then

$$\frac{dy}{dx} = \frac{(x + dx)^2 - x^2}{dx}$$

After squaring and cancelling we get,  $\frac{dy}{dx} = 2x + dx \approx 2x$  therefore

$$\circ\left(\frac{dy}{dx}\right) = 2x.$$

That is, the function  $x^2$  is differentiable with derivative  $2x$ .

Example 9.2. Let's see how to prove the Chain Rule! Suppose  $f(x)$  and  $g(x)$  are differentiable at the appropriate places and we wish to show

that the function  $h(x) = f(g(x))$  is differentiable with derivative  $h'(x) = f'(g(x))g'(x)$ . For any non-zero infinitesimal  $dx$ , write  $dg = g(x+dx) - g(x)$  and  $dh = h(x+dx) - h(x)$  then

$$dh = f(g(x+dx)) - f(g(x)) = f(g(x) + dg) - f(g(x)).$$

We want to show that for any non-zero infinitesimal  $dx$ ,

$$(1) \quad \frac{dh}{dx} \approx f'(g(x))g'(x).$$

Let non-zero infinitesimal  $dx$  be given. By continuity of  $g(x)$ ,  $dg$  is also infinitesimal.

Case 1.  $dg = 0$ . Then  $dh = 0$ , so  $\circ(\frac{dg}{dx}) = g'(x) = 0$  and  $\frac{dh}{dx} = 0$ . Thus

both sides of (1) are zero, so (1) holds.

Case 2.  $dg \neq 0$ . Then  $\frac{dh}{dx} = \frac{dh}{dg} \cdot \frac{dg}{dx}$  that is

$$(2) \quad \frac{dh}{dx} = \frac{f(g(x)+dg) - f(g(x))}{dg} \cdot \frac{g(x+dx) - g(x)}{dx}.$$

The two factors of the right side of (2) are infinitely close to  $f'(g(x))$  and  $g'(x)$  respectively. Now using the rules given in Section 2 for manipulating the symbol  $\approx$  we get

$$\frac{dh}{dx} \approx f'(g(x)) \cdot g'(x)$$

as desired.

## 10. INTEGRATION

Let  $f(x)$  be a standard function integrable on the standard interval  $[a, b]$ . For each standard  $n$  let

$$a = a_0^n < a_1^n < \cdots < a_n^n = b$$

be a partition of the interval into  $n$  subintervals of equal length. The Riemann sums

$$S_n = \sum_{i=1}^n f(a_i^n) (a_i^n - a_{i-1}^n)$$