11. The Main Theorem Revisited

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **20 (1974)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **12.07.2024**

constitute an infinite sequence, and by the Main Theorem this sequence can be extended to a sequence defined on N^*. For an infinite natural number α it seems natural to denote the α^{th} term S_α by

$$\sum_{i=1}^{\alpha} f(a_i^\alpha)\,(a_i^\alpha - a_{i-1}^\alpha)\,. \tag{1}$$

We might think of this as a "Riemann sum" on an infinitely fine net. The use of $\sum$ notation seems appropriate because the "sum" shares (by virtue of the Main Theorem) many properties of standard finite sums. For example, the property (omitting the summands for brevity)

$$\sum_{i=1}^{\alpha} = \sum_{i=1}^{\beta} + \sum_{i=\beta+1}^{\alpha}\,.$$

Now since

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(a_i^n)\,(a_i^n - a_{i-1}^n) = \int_a^b f(x)\,dx\,,$$

we see, using the non-standard characterization of the notion of limit of a sequence, that if α is an infinite natural number

$$\sum_{i=1}^{\alpha} f(a_i^\alpha)\,(a_i^\alpha - a_{i-1}^\alpha) \approx \int_a^b f(x)\,dx\,.$$

A further development of the theory of Integration and in particular a non-standard characterization of the Riemann integrable functions requires more machinery than we are prepared to set up here.

11. The Main Theorem Revisited

The version of the Main Theorem which we gave you in Section 3 is a specialization of a considerably more general result. While we stated it in terms of the real number system R, it happens to be true of any non-empty set X whatsoever. This opens the way for a penetration of the methods of Non-Standard Analysis to other branches of mathematics. For example, one might extend the complex number system C to a field C^*. There one could have "polygons" with sides of infinitely small length and vertices indexed by the initial segment of N^* determined by some infinite natural number.

What is also true is that R^* is even more closely related to R than we've suggested so far. Our version of the Main Theorem didn't permit as admissible statements those which had variables ranging over the functions on R, the relations on R or the subsets of R. A generalization of the theorem to include such statements is impossible. If it were possible, the Axiom of Completeness would be admissible and we'd have that R^* is complete, which contradicts a result seen previously. It turns out, however, that there exists a distinguished class of functions on R^*, a distinguished class of relations on R^*, and a distinguished class of subsets of R^* such that all statements with function, relation, and set variables can now be allowed in applications of the theorem provided that in R^* these variables are constrained to vary only over these distinguished classes. Robinson calls the functions, relations, and subsets of R^* in these classes *internal* functions, relations and subsets. Expressed differently what we are saying is that if you wore spectacles which were opaque to all functions, relations, and subsets of R^* other than the internal ones, you'd swear that R^* is complete, N^* is well ordered, etc. Your glasses wouldn't let you see the counter-examples! What is remarkable is that *one* pair of spectacles can be made to work for *all* the new statements. If only the Axiom of Completeness were at issue, we could simply choose a pair of spectacles which blocks out the bounded subsets of R^* which don't have least upper bounds. Using the improvement of the Main Theorem just mentioned the Theory of Integration, for example, becomes more susceptible to the methods of Non-Standard Analysis, and some of the argumentation elsewhere in this article could be simplified.

Conclusion

At the turn of the century Bertrand Russell wrote:

"... hence infinitesimals as explaining continuity must be regarded as unnecessary, erroneous, and self contradictory."

This remark gives some indication of the degree of disrepute into which the use of infinitesimals had fallen, and it serves to underscore the achievement in its eventual vindication by Robinson. Russell's work in logic, it should be mentioned, constituted one of the important steps along the way. Such is the unexpected path the development of ideas sometimes follows!