AN IMMERSION OF $T^n - D^n$ INTO R^n

by Steven FERRY

Let $T^n = S^1 \times ... \times S^1$ be the n-torus and let $D^n \subset T^n$ be an embedded disc. Kirby, Siebenmann, and Edwards use immersions of $T^n - D^n$ into R^n repeatedly in their work on stable homeomorphisms and triangulations. Although the existence of such immersions follows trivially from the work of Smale and Hirsch, it is appealing to have a more elementary construction. We will provide an explicit formula. Less explicit constructions have been given by D. Barden and J. Milnor.

Elements of T^n will be written as $(\theta_1, ..., \theta_n)$ with $\theta_1 \in S^1$. θ_1 will also be thought of as a real number $0 \leqslant \theta < 2\pi$. Occasionally $(\theta_1, ..., \theta_n)$ will be denoted by θ.

LEMMA 1. Let $(\theta, t) \to \big(f_1(\theta, t), ..., f_{n+1}(\theta, t)\big)$ be an embedding of $T^n \times (-1, 1)$ into R^{n+1} such that $f_{n+1}(\theta, t) > 0$. The map of $T^{n+1} \times (-1, 1)$ into R^{n+2} defined by

$$(\theta, t) \to \big(f_1(\theta, t), ..., f_n(\theta, t), f_{n+1}(\theta, t) \cos \theta_{n+1}, f_{n+1}(\theta, t) \sin \theta_{n+1}\big)$$

is an embedding.

We define the *standard embedding* of $T^n \times (-1, 1)$ into R^{n+1} to be the embedding obtained by starting with $(\theta_1, t) \to \big((1+t) \cos \theta_1, (1+t) \sin \theta_1 + 2\big)$ and iterating the process described in lemma 1. At each stage we must add 2^n to the last term so that the condition $f_{n+1}(\theta, t) > 0$ will be satisfied. For example, in the standard embedding of $T^3 \times (1, 1) \to R^4$ we have

$$f_3(\theta, t) = \big(\big((1+t) \sin \theta_1 + 2\big) \sin \theta_2 + 4\big) \cos \theta_3$$

and

$$f_4(\theta, t) = \big(\big((1+t) \sin \theta_1 + 2\big) \sin \theta_2 + 4\big) \sin \theta_3 + 8 .$$

Let $S = \{ \theta \in T^n \mid \theta_1 = 0 \text{ for some } i, 1 \leqslant i \leqslant n \}$.
Let $\varphi: T^n \to R^1$ be defined by

$$\varphi(\theta) = \frac{\sin \theta_1 ... \sin \theta_n}{2^n} + \frac{\sin \theta_2 ... \sin \theta_n}{2^{n-1}} + ... + \frac{\sin \theta_n}{2} .$$

Theorem 1. Let $(\theta, t) \to (f_1(\theta, t), ..., f_{n+1}(\theta, t))$ be the standard embedding of $T^n \times (-1, 1)$ into R^{n+1}. For some $\varepsilon > 0$ the map $\theta \to$

$\rightarrow (f_1(\theta, \varepsilon\varphi(\theta)), \ldots, f_n(\theta, \varepsilon\varphi(\theta)))$ has nonsingular Jacobian on S. It therefore immerses a regular neighborhood of S (i.e. $T^n - D^n$) into R^n.

Proof. Using elementary properties of determinants, we compute:

$$\det\left(\frac{\partial f_i}{\partial \theta_j} + \varepsilon \frac{\partial f_i}{\partial t}\frac{\partial \varphi}{\partial \theta_j}\right)\Bigg|_{\substack{\theta \in S \\ t = \varepsilon\varphi}} = \det\left(\begin{array}{c|c} \dfrac{\partial f_i}{\partial \theta_j} + \varepsilon \dfrac{\partial f_i}{\partial t}\dfrac{\partial \varphi}{\partial \theta_j} & 0 \\ \hline \dfrac{\partial \varphi}{\partial \theta_j} & 1 \end{array}\right)\Bigg|_{\substack{\theta \in S \\ t = \varepsilon\varphi}} =$$

$$= \det\left(\begin{array}{c|c} \dfrac{\partial f_i}{\partial \theta_j} & -\varepsilon \dfrac{\partial f_i}{\partial t} \\ \hline \dfrac{\partial \varphi}{\partial \theta_j} & 1 \end{array}\right)\Bigg|_{\substack{\theta \in S \\ t = \varepsilon\varphi}} = \det\left(\begin{array}{c|c} \dfrac{\partial f_i}{\partial \theta_j} & -\varepsilon \dfrac{\partial f_i}{\partial t} \\ \hline \dfrac{\partial \varphi}{\partial \theta_j} & 0 \end{array}\right)\Bigg|_{\substack{\theta \in S \\ t = \varepsilon\varphi}} + \det\left(\begin{array}{c|c} \dfrac{\partial f_i}{\partial \theta_j} & -\varepsilon \dfrac{\partial f_i}{\partial t} \\ \hline 0 & 1 \end{array}\right)\Bigg|_{\substack{\theta \in S \\ t = \varepsilon\varphi}}$$

By construction, f_i involves only $\theta_1, \ldots, \theta_i$ and $\dfrac{\partial f_i}{\partial \theta_i}$ has a factor of $\sin \theta_i$. Thus, on S the upper left hand corner of the second matrix is triangular with at least one zero on the diagonal. We have

$$\det\left(\frac{\partial f_i}{\partial \theta_j} + \frac{\partial f_i}{\partial t}\frac{\partial \varphi}{\partial \theta_j}\right)\Bigg|_{\substack{\theta \in S \\ t = \varepsilon\varphi}} = -\varepsilon \det\left(\begin{array}{c|c} \dfrac{\partial f_i}{\partial \theta_j} & \dfrac{\partial f_i}{\partial t} \\ \hline \dfrac{\partial \varphi}{\partial \theta_j} & 0 \end{array}\right)\Bigg|_{\substack{\theta \in S \\ t = \varepsilon\varphi}}$$

Notice that $f_{n+1}(\theta, 0) = 2^n \varphi(\theta)$ and that $\dfrac{\partial f_{n+1}}{\partial t}$ is identically zero on S. Thus, if the above determinant is evaluated at $\theta \in S$, $t = 0$ it is $\left(\dfrac{-\varepsilon}{2^n}\right)$ times the determinant of the Jacobian of the standard embedding. It is therefore nonsingular when evaluated at $\theta \in S$, $t = \varepsilon\varphi$ for sufficiently small ε. This completes the proof.

In essence, we have perturbed the image of $T^n \times 0$ in R^{n+1} along its normal bundle so that projection into R^n is an immersion on S. More precise calculations show that ε may be taken to be 1.

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