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NOTES ON THE CONGRUENCE $y^2 \equiv x^5 - a \pmod{p}$

by A. R. RAJWADE

1. INTRODUCTION

In a previous paper [3] we proved the following

THEOREM. *Let $p \equiv 1 \pmod{5}$ be a rational prime and g a fixed primitive root mod p . Then the number of solutions of the congruence*

$$(1) \quad y^2 \equiv x^5 - a \pmod{p}$$

is $p + \Delta_a$, where Δ_a is equal to ¹⁾

$$(2) \quad \left(\frac{-4a}{\pi_1}\right)_{10} \cdot \pi_3 \pi_4 + \left(\frac{-4a}{\pi_2}\right)_{10} \cdot \pi_1 \pi_3 \\ + \left(\frac{-4a}{\pi_3}\right)_{10} \cdot \pi_2 \pi_4 + \left(\frac{-4a}{\pi_4}\right)_{10} \cdot \pi_1 \pi_2 .$$

Here $p = \pi_1 \pi_2 \pi_3 \pi_4 = \pi_1 \cdot \sigma \pi_1 \cdot \sigma^3 \pi_1 \cdot \sigma^2 \pi_1$, with $\sigma: \zeta \rightarrow \zeta^2$, is the decomposition of p in $Q(\zeta)$, $\zeta^5 = 1$, $\zeta \neq 1$ and π_1 is chosen to satisfy $(g/\pi_1)_5 = \zeta$, so that $(g/\pi_i)_5 = \zeta^i$, and the π_j are normalized so that the products $S = \pi_1 \pi_2$, $\bar{S} = \pi_3 \pi_4$, $T = \pi_1 \pi_3$, $\bar{T} = \pi_2 \pi_4$ (all polynomials in ζ) satisfy

1. $S(\zeta) \cdot S(\zeta^{-1}) \equiv [S(1)]^2 \pmod{5}$,
2. $S(\zeta) \equiv S(1) \pmod{(1-\zeta)^2}$,
3. $S(1) \equiv 4 \pmod{5}$.

(and similarly for \bar{S}, T, \bar{T}).

In (2) the 4 products $\pi_i \pi_j$ are those 4 out of the 6 combinations $\pi_1 \pi_2, \pi_1 \pi_3, \pi_1 \pi_4, \pi_2 \pi_3, \pi_2 \pi_4, \pi_3 \pi_4$ for which $\bar{\pi}_i \neq \pi_j$. But there is no symmetrical way of coupling the residue symbol $\left(\frac{-4a}{\pi_i}\right)_{10}$ with $\pi_j \pi_k$. We ask: What do other expressions similar to Δ_a represent? For example the expression

¹⁾ See Appendix for the definitions of $(\alpha' \beta)_{10}, (\alpha' \beta)_5, (a/p)_Z$.

$$\left(\frac{-4a}{\pi_1}\right)_{10} \cdot \pi_1 \pi_2 + \left(\frac{-4a}{\pi_2}\right)_{10} \cdot \pi_2 \pi_4 + \left(\frac{-4a}{\pi_3}\right)_{10} \cdot \pi_1 \pi_3 + \left(\frac{-4a}{\pi_4}\right)_{10} \cdot \pi_3 \pi_4$$

being the trace of $(-4a/\pi_1)_{10} \cdot \pi_1 \pi_2$, is a rational integer. What does it represent?

One could also remove the various restrictions on the π_i in the expression for Δ_a and ask what Δ_a then represents. The object of this note is to answer these questions and also to determine the set $\{\Delta_a \mid a = 1, 2, 3, \dots, p - 1\}$.

It is immediate that Δ_a can take only 10 distinct values. This follows by looking at (2) or directly from the congruence (1) as follows: Let $(e, p) = 1$, then we have

$$\Delta_a = \sum \left(\frac{x^5 - a}{p} \right) \text{ and so } \Delta_{ae} 5 = (e/p)_Z \cdot \Delta_a.$$

It follows that the distinct values taken by the Δ_a , for $a = 1, 2, \dots, p - 1$ are just $\pm \Delta_g, \pm \Delta_{g^2}, \pm \Delta_{g^3}, \pm \Delta_{g^4}, \pm \Delta_{g^5}$. We shall determine these 10 values as a set. Which value is associated with which a will not be clear except when $4a$ is a quintic residue mod p .

2. DETERMINATION OF Δ_a

WITHOUT THE NORMALIZATION RESTRICTIONS ON THE π_j

Write $p = \pi \cdot \pi^\sigma \cdot \pi^{\sigma^3} \cdot \pi^{\sigma^2}$ (with $(g/\pi)_5 = \zeta) = \pi_1 \pi_2 \pi_3 \pi_4$ say. Since the restrictions on π are going to be removed, we denote Δ_a by $\Delta_a(\pi)$. We write (2) in a more convenient form viz

$$(3) \quad \Delta_a(\pi) = \left(\frac{-a}{p}\right)_Z \cdot \left[\left(\frac{4a}{\pi_1}\right)_5 \cdot \pi_1 \pi_3 + \left(\frac{4a}{\pi_2}\right)_5 \cdot \pi_1 \pi_2 + \left(\frac{4a}{\pi_3}\right)_5 \cdot \pi_3 \pi_4 + \left(\frac{4a}{\pi_4}\right)_5 \cdot \pi_2 \pi_4 \right].$$

Thus $\Delta_a(\pi) = \text{Tr} [(-a/p)_Z (4a/\pi)_5 \pi \pi^{\sigma^3}]$.

Let the condition $(g/\pi)_5 = \zeta$ be retained first so that we only change π to an associate $\eta \pi$ where $\eta = \zeta^i \varepsilon$ ($0 \leq i \leq 4$) with ε a real fundamental unit, say $\pm \left(\frac{1 + \sqrt{5}}{2}\right)^j$, $j \in \mathbf{Z}$, of $Q(\sqrt{5})$. We have the following

THEOREM 1. $\Delta_a(\zeta^i \varepsilon \cdot \pi) = \Delta_{ab}(\pi)$ where $(b/\pi)_5 = \zeta^{5-i}$ and $(b/p)_Z \neq N_{Q(\sqrt{5})/Q}(\varepsilon)$.