

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 21 (1975)
Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: NOTES ON THE CONGRUENCE $y^2 \equiv x^5 - a \pmod{p}$
Autor: Rajwade, A. R.
Kapitel: 3. The restriction $(g/\pi)_5 = \zeta$ removed
DOI: <https://doi.org/10.5169/seals-47329>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. [Siehe Rechtliche Hinweise.](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. [Voir Informations légales.](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. [See Legal notice.](#)

Download PDF: 22.12.2024

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Proof. Step 1.

$$\begin{aligned}\Delta_a(\zeta\pi) &= \text{Tr} [(-a/p)_Z (4a/\zeta\pi)_5 (\zeta\pi) (\zeta\pi)^{\sigma^3}] \\ &= \text{Tr} [(-a/p)_Z (4a/\pi)_5 \cdot \zeta^4 \cdot \pi\pi^{\sigma^3}] \\ &= \text{Tr} [(-au/p)_Z (4au/\pi)_5 \cdot \pi\pi^{\sigma^3}],\end{aligned}$$

where $(u/p)_Z = 1$, $(u/\pi)_5 = \zeta^4$, and this $= \Delta_{au}(\pi)$. It follows that $\Delta_a(\zeta^i\pi) = \Delta_{au}(\pi)$, where $(u/p)_Z = 1$ and $(u/\pi)_5 = \zeta^{5-i}$ ($i=0, 1, 2, 3, 4$).

Step 2.

$$\begin{aligned}\Delta_a(\varepsilon\pi) &= \text{Tr} [(-a/p)_Z (4a/\varepsilon\pi)_5 \cdot \varepsilon\pi \cdot (\varepsilon\pi)^{\sigma^3}] \\ &= \text{Tr} [(-a/p)_Z (4a/\pi)_5 \cdot N_{Q(\sqrt{5})/Q}(\varepsilon) \cdot \pi\pi^{\sigma^3}] \\ &= \Delta_{av}(\pi),\end{aligned}$$

where $(v/p)_Z = N_{Q(\sqrt{5})/Q}(\varepsilon)$, $(v/\pi)_5 = 1$.

Combining steps 1 and 2 we get:

$$\begin{aligned}\Delta_a(\zeta^i\varepsilon\pi) &= \Delta_{au}(\varepsilon\pi) \text{ where } (u/p)_Z = 1, (u/\pi)_5 = \zeta^{5-i} \\ &= \Delta_{au.v}(\pi) \text{ where } (v/p)_Z = \text{Norm } \varepsilon, (v/\pi)_5 = 1, \\ &= \Delta_{ab}(\pi) \text{ where } b = uv \text{ satisfies the conditions of}\end{aligned}$$

theorem 1. This completes the proof of theorem 1.

We next remove the restriction $(g/\pi)_5 = \zeta$ and see what the Δ_a 's mean then.

3. THE RESTRICTION $(g/\pi)_5 = \zeta$ REMOVED

Here we have to look at $\Delta_a(\pi^\sigma)$ (and similarly $\Delta_a(\pi^{\sigma^2})$ and $\Delta_a(\pi^{\sigma^3})$). We have the following

THEOREM 2. $\Delta_a(\pi^\sigma) = \Delta_a(\pi)$.

Proof. $\Delta_a(\pi^\sigma) = \text{Tr} [(-a/p)_Z (4a/\pi^\sigma)_5 \cdot \pi^\sigma \cdot (\pi^\sigma)^{\sigma^3}]$.

Now $(4a/\pi^\sigma)_5 = (4a/\pi_2)_5$, and if $4a \equiv g^v \pmod{p}$ then this $= (g^v/\pi_2)_5 = (g/\pi_2)_5^v = \zeta^{2v} = (g^v/\pi_1)_5^2 = (4a/\pi_1)_5^2 = \sigma[(4a/\pi)_5]$. Hence

$$\begin{aligned}\Delta_a(\pi^\sigma) &= \text{Tr} [(-a/p)_Z \cdot \sigma(4a/\pi)_5 \cdot \pi \cdot \pi^{\sigma^3}] \\ &= \text{Tr} [\sigma((-a/p)_Z (4a/\pi)_5 \cdot \pi\pi^{\sigma^3})] \\ &= \Delta_a(\pi) \text{ as required.}\end{aligned}$$

A clearer insight is gained into this by looking at the whole thing directly as follows.

Since the choice of g is arbitrary, we change g to another primitive root g^r with $(r, p-1) = 1$, $r = i \pmod{5}$, $i = 1, 2, 3, 4$. This does not alter Δ_a (as Δ_a is independent of g) but replaces π by any desired π_i so that $\Delta_a(\pi) = \Delta_a$ (any other π). Note that such an r exists, for all we want is, for $i = 1, 2, 3, 4$, a λ such that $(i+5\lambda, p-1) = 1$. Now $i+5\lambda$ takes infinitely many prime values as λ takes positive integer values since $(i, 5) = 1$; so λ may be chosen so that $i+5\lambda$ is a prime avoiding the primes occurring in $p-1$.

4. EXPRESSIONS ALLIED TO $\Delta_a(\pi)$

We fix our π now with $(g/\pi)_5 = \zeta$ and normalize it too. It is clear that there are only 3 expressions allied to $\Delta_a(\pi)$ viz $(-a/p)_Z (4a/\pi)_5 \cdot \pi \cdot \pi^\sigma +$ conjugates, $(-a/p)_Z (4a/\pi)_5 \cdot \pi^\sigma \cdot \pi^{\sigma^2} +$ conjugates and $(-a/p)_Z (4a/\pi)_5 \cdot \pi^{\sigma^2} \cdot \pi^{\sigma^3} +$ conjugates. This is so because changing the first term of $\Delta_a(\pi)$ fixes the changes in the other terms (otherwise we will not even get a rational integer!). Let us look at the first of these (the others would be similar), which equals $\text{Tr} [(-a/p)_Z (4a/\pi)_5 \cdot \pi \pi^\sigma]$. We have the following theorem:

THEOREM 3. $\text{Tr} [(-a/p)_Z (4a/\pi)_5 \cdot \pi \pi^\sigma] = \Delta_{au} - 1(\pi)$, where $(u/p)_Z = 1$ and $(u/\pi)_5 = (4a/\pi)_5$.

Proof. We have

$$\begin{aligned} \Delta_a(\pi) &= \text{Tr} [(-a/p)_Z (4a/\pi)_5 \cdot \pi \cdot \pi^{\sigma^3}] \\ &= \text{Tr} [(-a/p)_Z (4a/\pi^\sigma)_5 \cdot \pi^\sigma \cdot \pi^{\sigma^3}] \text{ by 3 on letting } \pi \rightarrow \pi^\sigma, \\ &= \text{Tr} [(-a/p)_Z (16a^2/\pi)_5 \cdot \pi^\sigma \cdot \pi] \text{ since } (4a/\pi^\sigma)_5 = (g^v/\pi_2)_5 \\ &= (g^v/\pi_1)_5^2 = (4a/\pi)_5^2 = (16a^2/\pi)_5, \\ &= \text{Tr} [(-au/p)_Z (4(au)/\pi)_5 \cdot \pi \pi^\sigma], \text{ where } (u/p)_Z = 1 \text{ and } (u/p)_5 \\ &= (4a/\pi)_5. \end{aligned}$$

Now writing a for au we get the theorem.

It follows that the expressions allied to $\Delta_a(\pi)$ also represent the number of solutions of the congruence (1) for a suitable value of a .

5. THE SET $\{\Delta_a \mid a = 1, 2, 3, \dots, p-1\}$

Dickson's paper on cyclotomy [1] includes the following Theorem (theorem 8 of [1]). Let $p \equiv 1 \pmod{5}$ be a rational prime. Then the Diophantine equations