2. Zeros of Exponential Polynomials

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As Professor Turán pointed out to us, one source of information on the problem is the papers of J. F. Ritt. Ritt provides a factorisation theory for exponential polynomials [10], and shows *inter alia* that if a quotient of exponential polynomials is an entire function then it is an exponential polynomial [11]. We describe these results in section 3.

In the case where all the frequencies of one of the exponential polynomials are rational we confirm that there is indeed a common factor. Even this special case seems to require a non-trivial argument; we employ the theorem of Skolem-Mahler-Lech on recurrence sequences with infinitely many vanishing terms (Lech [4], Mahler [6]). Conversely we observe in section 4 that an affirmative answer to the problem implies a generalised form of the Skolem-Mahler-Lech theorem. For a similar application of this theorem to zeros of exponential polynomials see Jager [3].

It follows from the results mentioned in sections 3 and 4 that one can define the greatest common divisor $h \in E$ of two exponential polynomials $f, g \in E$. An affirmative answer to the problem then implies that the set of zeros of the gcd h is all but at most finitely many of the common zeros of f and g. We make these and other remarks in section 5. We conclude this section with an example due to Montgomery which shows that "approximate methods" in the obvious manner are doomed to failure.

2. Zeros of Exponential Polynomials

Given an exponential polynomial,

$$f(z) = \Sigma a_j e^{\alpha_j z} = a_1 e^{\alpha_1 z} + \ldots + a_n e^{\alpha_n z}$$

denote by C_f the convex polygon in the complex plane defined by the complex conjugates of the frequencies; that is, the convex hull of the points $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n$. Then *the zeros of f lie in half-strips in the directions of the exterior normals to C_f*. More quantitatively, *suppose an edge of the polygon C_f has length l. Then the number of zeros of $f(z)$ in the half-strip perpendicular to that edge and of absolute value less than R is*

$$(2) \qquad \frac{lR}{2\pi} + O(1)$$

; see Pólya [8], D. G. Dickson [2].

It can also be shown that near every line in and parallel to the sides of a strip of zeros lie infinitely many zeros of the exponential polynomial, see Moreno [7], van der Poorten [9]. From a different point of view, one can obtain upper bounds for the number of zeros of $f(z)$ in any circle of radius R in the complex plane; if $\Delta = \max_k |\alpha_k|$, then such a bound is

$$3(n-1) + 4R\Delta \qquad ; \text{ see Tijdeman [17].}$$

For further details the reader is referred to the cited articles and the appropriate references mentioned therein.

3. Factorisation of Exponential Polynomials

We describe the results of J. F. Ritt [10], [11].

Define an ordering on the set of complex numbers by: $\alpha < \beta$ if $\mathcal{R}e\,\alpha \leqslant \mathcal{R}e\,\beta$, and if $\mathcal{R}e\,\alpha = \mathcal{R}e\,\beta$ then $\mathcal{I}m\,\alpha < \mathcal{I}m\,\beta$. We will suppose in the sequel, that, unless indicated otherwise, any exponential polynomial

$$\Sigma\, a_j\, e^{\alpha_j z} = a_1\, e^{\alpha_1 z} + \ldots + a_n\, e^{\alpha_n z}$$

is so normalised that $a_1 = 1$ and $0 = \alpha_1 < \alpha_2 < \ldots < \alpha_n$. Of course the normalisation is effected by multiplying by some unit $be^{\beta z}$, $b \neq 0$ thus not affecting the zeros of the exponential polynomial. Many of the remarks below are invalid if the normalisation is not assumed.

Ritt [10] firstly shows that *if the exponential polynomial* $\sum b_j e^{\beta_j z}$ *divides the exponential polynomial* $\sum a_j e^{\alpha_j z}$ *(in the ring E of exponential polynomials) then the frequencies* $\beta_1, \ldots, \beta_m$ *are linear combinations with rational coefficients of the frequencies* $\alpha_1, \ldots, \alpha_n$. Now, following Ritt, call an exponential polynomial $\sum a_j e^{\alpha_j z}$ *simple* if its frequencies are commensurable, that is, there is a minimal (in the sense of the ordering on **C**) number α such that each α_j is a non-negative integer multiple of α. So such a simple exponential polynomial $f(z)$ is a polynomial in $e^{\alpha z}$ and factorises into a finite product of functions of the shape $1 + ae^{\alpha z}$, $a \in \mathbf{C}$. Of course, $1 + ae^{\alpha z}$ has factors of the shape $1 + a' e^{(\alpha/m)z}$ for each $m = 1, 2, 3, \ldots$ but it follows from Ritt's lemma mentioned above that every factor of $f(z)$ is a product of such factors. Similarly, call an exponential polynomial *irreducible* if it has no non-trivial (that is, other than units and associates) factors in the ring E. Then Ritt's principal result is that *an exponential polynomial can be factorised uniquely as a finite product of simple exponential polynomials such that their sets of frequencies are pairwise incommensurable, and a finite product of irreducible exponential polynomials.*

We outline the structure of the proof. Firstly one shows that there exist complex numbers $\mu_1, \mu_2, \ldots, \mu_p$ linearly independent over the field of