3. Factorisation of Exponential Polynomials

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obtain upper bounds for the number of zeros of $f(z)$ in any circle of radius R in the complex plane; if $\Delta = \max_k |\alpha_k|$, then such a bound is

$$3(n-1) + 4R\Delta \qquad ; \text{ see Tijdeman [17].}$$

For further details the reader is referred to the cited articles and the appropriate references mentioned therein.

3. Factorisation of Exponential Polynomials

We describe the results of J. F. Ritt [10], [11].

Define an ordering on the set of complex numbers by: $\alpha < \beta$ if $\mathcal{R}e\,\alpha \leqslant \mathcal{R}e\,\beta$, and if $\mathcal{R}e\,\alpha = \mathcal{R}e\,\beta$ then $\mathcal{I}m\,\alpha < \mathcal{I}m\,\beta$. We will suppose in the sequel, that, unless indicated otherwise, any exponential polynomial

$$\Sigma\, a_j\, e^{\alpha_j z} = a_1\, e^{\alpha_1 z} + \dots + a_n\, e^{\alpha_n z}$$

is so normalised that $a_1 = 1$ and $0 = \alpha_1 < \alpha_2 < \dots < \alpha_n$. Of course the normalisation is effected by multiplying by some unit $be^{\beta z}$, $b \neq 0$ thus not affecting the zeros of the exponential polynomial. Many of the remarks below are invalid if the normalisation is not assumed.

Ritt [10] firstly shows that *if the exponential polynomial* $\sum b_j e^{\beta_j z}$ *divides the exponential polynomial* $\sum a_j e^{\alpha_j z}$ *(in the ring E of exponential polynomials) then the frequencies* $\beta_1, \dots, \beta_m$ *are linear combinations with rational coefficients of the frequencies* $\alpha_1, \dots, \alpha_n$. Now, following Ritt, call an exponential polynomial $\sum a_j e^{\alpha_j z}$ *simple* if its frequencies are commensurable, that is, there is a minimal (in the sense of the ordering on **C**) number α such that each α_j is a non-negative integer multiple of α. So such a simple exponential polynomial $f(z)$ is a polynomial in $e^{\alpha z}$ and factorises into a finite product of functions of the shape $1 + ae^{\alpha z}$, $a \in \mathbf{C}$. Of course, $1 + ae^{\alpha z}$ has factors of the shape $1 + a' e^{(\alpha/m)z}$ for each $m = 1, 2, 3, \dots$ but it follows from Ritt's lemma mentioned above that every factor of $f(z)$ is a product of such factors. Similarly, call an exponential polynomial *irreducible* if it has no non-trivial (that is, other than units and associates) factors in the ring E. Then Ritt's principal result is that *an exponential polynomial can be factorised uniquely as a finite product of simple exponential polynomials such that their sets of frequencies are pairwise incommensurable, and a finite product of irreducible exponential polynomials.*

We outline the structure of the proof. Firstly one shows that there exist complex numbers $\mu_1, \mu_2, \dots, \mu_p$ linearly independent over the field of rational numbers $\mathbf{Q}$ such that each frequency of f is a linear combination of the $\mu_1, \ldots, \mu_p$ with non-negative integer coefficients. It follows that any normalised factor (in the ring E) of f similarly has frequencies which are linear combinations of the $\mu_1, \ldots, \mu_p$ with non-negative rational coefficients. Now write $y_1 = e^{\mu_1 z}, \ldots, y_p = e^{\mu_p z}$. Then $f(z)$ becomes a polynomial $q(y_1, \ldots, y_p) = q(y)$. Moreover, it is clear that for each finite factorisation of $f(z)$ in the ring E there is, for some set of positive integers $t_1, \ldots, t_p$ a factorisation of $q(y_1^{t_1}, \ldots, y_p^{t_p}) = q(y^t)$ in the ring $\mathbf{C}[y_1, \ldots, y_p] = \mathbf{C}[y]$; conversely to each such factorisation in $\mathbf{C}[y]$ into polynomials with constant term 1, there is a factorisation in E. We suppose henceforth that polynomials have constant term 1. To make the correspondence, observed above, one-one, Ritt defines $q(y) \in \mathbf{C}[y]$ to be *primary* if for each $i = 1, 2, \ldots, p$ the exponents of y_i in the monomials comprising $q(y)$ have greatest common divisor 1. One sees that if $q(y)$ is primary then of the irreducible factors of $q(y^t)$ in $\mathbf{C}[y]$ either all or none are again primary. Then if $f(z)$ is represented by a primary polynomial $q(y)$ each finite factorisation of $f(z)$ in E corresponds one-one to a minimal choice of $t = (t_1, \ldots, t_p)$ and a factorisation of $q(y^t)$. Ritt now shows that if $q(y)$ is primary and has more than two terms (including constant term 1) then there are only finitely many sets $t = (t_1, \ldots, t_p)$ of positive integers such that the irreducible factors of $q(y^t)$ are primary. This settles the finiteness of the factorisation and the remainder of the proof is straightforward.

In [11] Ritt proves that *if a quotient of exponential polynomials is an entire function then it is an exponential polynomial;* in [12] it is shown *inter alia* that it is sufficient that the quotient be regular in a sector of opening greater than π. We remark on generalisations of Ritt's result in section 5. An equivalent assertion to Ritt's theorem is *if every zero of $f(z) = \sum a_j e^{\alpha_j z}$ is a zero of $g(z) = \sum b_j e^{\beta_j z}$ then $f(z)$ divides $g(z)$ in the ring E.* The principle of the proof is as follows: denote by $|C_f|$ the maximal real cross-section, that is, parallel to the real axis, of the polygon C_f defined in section 2. Then one shows there exist exponential polynomials q and r such that $g = qf + r$ and $|C_r| < |C_f|$. It follows that r has less zeros than does f in sufficiently large rectangles, whence $r \equiv 0$ as required.

We should remark that by a different method Allen Shields [14] has shown that a quotient of exponential polynomials is an exponential polynomial already provided that the number of poles of the quotient in $|z| < R$ is $o(R)$. This result follows from the proof outlined above. We further note that H. N. Shapiro [19, §5] has given a division theorem related to Ritt's theorem.

The factorisation theory implies that we need consider only two cases of the Shapiro problem. Namely, firstly the case where at least one of the exponential polynomials f, g is simple. We settle this case, in the affirmative in section 4. Secondly one must take the case where at least one of the exponential polynomials is irreducible. Then an affirmative answer to the problem is equivalent to the truth of the following conjecture:

Let f, g be exponential polynomials and let f be irreducible. Then if f and g have infinitely many zeros in common, f divides g in the ring E (equivalently, g/f is an entire function).

Equivalent to this conjecture is: *if f, g are distinct irreducible exponential polynomials then f and g have at most finitely many common zeros.* This last formulation can be rephrased in terms of polynomials (with constant term 1):

Let $f(y) = f(y_1, \ldots, y_p)$, $g(y)$ be distinct polynomials irreducible in $\mathbf{C}[y]$ in the strong sense that for all sets $t_1, \ldots, t_p$ of positive integers, the polynomials $f(y^t) = f(y_1^{t_1}, \ldots, y_p^{t_p})$, $g(y^t)$ are so irreducible. Denote by $V \subset \mathbf{C}^p$ the set of common zeros of $f(y)$ and $g(y)$. Let $\mu_1, \ldots, \mu_p$ be numbers linearly independent over $\mathbf{Q}$. Then the curve $\{(e^{\mu_1 z}, \ldots, e^{\mu_p z}) : z \in \mathbf{C}\}$ meets V in at most finitely many points.

4. The Theorem of Skolem-Mahler-Lech

The following result was proved by Skolem [15] for the field of rational numbers, by Mahler [5] for the field of algebraic numbers, and by Lech [4] and Mahler [6] for arbitrary fields of characteristic zero (the assertion is false in fields of characteristic p):

Let $\{c_\nu\}$ be a sequence whose elements lie in a field of characteristic zero and satisfy a linear homogeneous recurrence relation

$$c_\nu = b_1 c_{\nu-1} + b_2 c_{\nu-2} + \ldots + b_n c_{\nu-n}, \qquad \nu = n, n+1, \ldots \tag{3}$$

Denote by $\mathrm{M} \subset \mathrm{N}$ *the set of indices ν such that $c_\nu = 0$. Then* M *is a finite union of arithmetic progressions* (the progressions may have common difference 0 and so consist of a single point). Hence those c_ν equal to zero occur periodically in the sequence from a certain index on.

It is well-known that there exist elements $\beta_1, \ldots, \beta_m$, namely the distinct zeros of the polynomial

$$z^n - b_1 z^{n-1} - b_2 z^{n-2} - \ldots - b_n, \tag{4}$$

and polynomials $p_1, \ldots, p_m$ where $1 + \deg p_j$ is the multiplicity of β_j as a zero of (4), $j = 1, 2, \ldots, m$, such that for all $\nu = 0, 1, 2, \ldots$