divisor of exponential polynomials f and g, then the set of zeros of h is all but at most finitely many of the common zeros of f and g. We have shown this to be the case if at least one of f and g is a simple exponential sum.

We see that a natural formulation of the Shapiro problem is: *If f and g are exponential polynomials, is it the case that there exists an exponential polynomial h, the set of zeros of which is exactly the set of common zeros of f and g?*

We recall that it is not, without qualification, the case that if every zero of $f \in E'$ is a zero of $g \in E'$ then f divides g in the ring E'; for example $(1-e^z)/z$ is not an element of E' (its set of integer zeros in not a finite union of arithmetic progressions). Equivalently, it follows that if $\Pi_{l=1}^{m}(e^{z/2^l}+1)$ divides an exponential polynomial $g(z)$ in the ring E' for all $m = 1, 2, \ldots$ then $1 - e^z$ divides $g(z)$ in E'.

The ideas we have mentioned attack an apparently analytic problem by essentially algebraic methods. Indeed, in a sense, "approximate" methods appear doomed to failure by virtue of the following proposition mentioned to the authors by H. L. Montgomery:

Proposition 3. *Let $\mu(\mathrm{r})$ be any positive-real-valued function decreasing to 0 as $\mathrm{r} \to \infty$. Then there exist exponential polynomials f, g such that for every $\mathrm{r}_0 > 0$ there is an $\mathrm{r} > \mathrm{r}_0$ and a $z \in \mathbf{C}$ with $\mathrm{r}_0 < |z| < \mathrm{r}$ such that $0 < |f(z) - g(z)| \leqslant \mu(\mathrm{r})$.*

Proof. Define an increasing sequence $\{n_l\}$ of integers by $n_0 = 0$ and $n_{l+1} - n_l \geqslant -\log(\mu(2^{n_l})/2\pi)/\log 2$ and write $\alpha = \sum_{l=0}^{\infty}(-1)^l 2^{-n_l}$. Let $f(z) = 1 - e^{2\pi i z}$ and $g(z) = 1 - e^{2\pi i \alpha z}$, and write $z_l = 2^{n_l}$, $l = 0, 1, 2, \ldots$. Then $f(z_l) = 0$ and $0 < |g(z_l)| = |1 - e^{2\pi i \alpha z_l}| = 2|\sin \pi \alpha z_l| \leqslant \mu(2^{n_l})$, as required. One notices that $f(z), g(z)$ have the property that there are infinitely many pairs z_l, z'_l with $f(z_l) = 0$, $g(z'_l) = 0$ and $|z_l - z'_l| \leqslant 2\mu(|z_l|)$.

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