7. SUMS OVER INTERVALS OF LENGTH k/8.

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class number formula for $S_{61}(\chi_{-d})$ is due to Lerch [44, p. 403], and those for $S_{62}(\chi_d)$ and $S_{63}(\chi_d)$ are also due to Lerch [44, p. 414]. In the terminology of class numbers, Holden [36] has established (6.4)-(6.6) in the associated special cases. Some results related to (6.1)-(6.3) were also found by Holden [39].

7. Sums over intervals of length $k/8$.

THEOREM 7.1. Let χ be even, let $\chi_{4k} = \chi_4\chi$, and let $\chi_{8k} = \chi_4\chi_8\chi$. Then

$$(7.1)\quad S_{81} = \frac{G(\chi)}{2\pi}\{\bar\chi(2)L(1,\bar\chi_{4k}) + 2^{1/2}L(1,\bar\chi_{8k})\},$$

$$S_{82} = \frac{G(\chi)}{2\pi}\{[2-\bar\chi(2)]L(1,\bar\chi_{4k}) - 2^{1/2}L(1,\bar\chi_{8k})\},$$

$$S_{83} = \frac{G(\chi)}{2\pi}\{-[2+\bar\chi(2)]L(1,\bar\chi_{4k}) + 2^{1/2}L(1,\bar\chi_{8k})\},$$

and

$$S_{84} = \frac{G(\chi)}{2\pi}\{\bar\chi(2)L(1,\bar\chi_{4k}) - 2^{1/2}L(1,\bar\chi_{8k})\}.$$

Let χ be odd and let $\chi_{8k} = \chi_8\chi$. Then

$$(7.2)\quad S_{81} = \frac{G(\chi)}{2\pi i}\left\{\left[2+\frac{1}{2}\bar\chi(4)\{1-\bar\chi(2)\}\right]L(1,\bar\chi) - 2^{1/2}L(1,\bar\chi_{8k})\right\},$$

$$S_{82} = \frac{G(\chi)}{2\pi i}\left\{\bar\chi(2)\left[1-\frac{3}{2}\bar\chi(2)+\frac{1}{2}\bar\chi(4)\right]L(1,\bar\chi) + 2^{1/2}L(1,\bar\chi_{8k})\right\},$$

$$S_{83} = \frac{G(\chi)}{2\pi i}\left\{\bar\chi(2)\left[-1+\frac{3}{2}\bar\chi(2)-\frac{1}{2}\bar\chi(4)\right]L(1,\bar\chi) + 2^{1/2}L(1,\bar\chi_{8k})\right\},$$

and

$$S_{84} = \frac{G(\chi)}{2\pi i}\left\{\left[2-\frac{1}{2}\bar\chi(4)\right][1-\bar\chi(2)]L(1,\bar\chi) - 2^{1/2}L(1,\bar\chi_{8k})\right\}.$$

We need only prove (7.1) and (7.2), for the remaining formulae can then be deduced from (7.1), (7.2), Theorem 3.2, Theorem 3.7, and elementary considerations. Since the proofs are similar to those in previous sections, we omit them. For the same reasons, proofs in sections 8-11 will not be given.

COROLLARY 7.2. If $d > 0$, we have

$$S_{81} > 0, \text{ if } \chi(2) = 1 \text{ or } 0;$$

$$S_{84} < 0, \text{ if } \chi(2) = -1 \text{ or } 0;$$

$$| S_{84} | < S_{81}, \text{ if } \chi(2) = 1;$$

$$| S_{81} | < - S_{84}, \text{ if } \chi(2) = -1;$$

$$S_{81} > S_{83}, \text{ if } \chi(2) = 1;$$

$$S_{82} > - S_{83}, \text{ if } \chi(2) = -1;$$

$$S_{82} < - S_{83}, \text{ if } \chi(2) = 1;$$

$$S_{82} > S_{84}, \text{ if } \chi(2) = -1;$$

$$S_{81} = S_{83}, \text{ if } \chi(2) = -1;$$

and

$$S_{82} = S_{84}, \text{ if } \chi(2) = 1.$$

If $d < 0$, we have

$$S_{82} > 0, \text{ if } \chi(2) = 1 \text{ or } 0;$$

$$S_{83} > 0;$$

$$S_{84} < 0, \text{ if } \chi(2) = 1;$$

$$| S_{82} | < S_{83}, \text{ if } \chi(2) = -1;$$

$$S_{81} > - S_{83};$$

$$S_{81} = - S_{82} = S_{84}, \text{ if } \chi(2) = -1;$$

and

$$S_{82} = S_{83} = - S_{84}, \text{ if } \chi(2) = 1.$$

Theorem 7.1 yields 8 formulae for class numbers. We shall list just those that we need to derive congruences.

COROLLARY 7.3. Let d be odd. If $d > 0$, then

$$S_{81}(\chi_d) = \frac{1}{4}\left(\frac{d}{2}\right) h(-4d) + \frac{1}{4} h(-8d) \tag{7.3}$$

and

$$S_{84}(\chi_d) = \frac{1}{4}\left(\frac{d}{2}\right) h(-4d) - \frac{1}{4} h(-8d). \tag{7.4}$$

If $d < 0$, then

(7.5) $$S_{81}(\chi_{-d}) = \frac{1}{4}\left\{5 - \left(\frac{d}{2}\right)\right\} h(d) - \frac{1}{4} h(8d),$$

(7.6) $$S_{83}(\chi_{-d}) = \frac{3}{4}\left\{1 - \left(\frac{d}{2}\right)\right\} h(d) + \frac{1}{4} h(8d),$$

and

(7.7) $$S_{84}(\chi_{-d}) = \frac{3}{4}\left\{1 - \left(\frac{d}{2}\right)\right\} h(d) - \frac{1}{4} h(8d).$$

Corollary 7.4. We have

$$h(-8p) \equiv h(-4p) \pmod 8, \text{ if } p \equiv 1,5 \pmod{16},$$
$$h(-8p) \equiv 4 + h(-4p) \pmod 8, \text{ if } p \equiv 9,13 \pmod{16},$$
$$h(-8p) \equiv 0 \pmod 8, \text{ if } p \equiv 15 \pmod{16},$$
$$h(-8p) \equiv 4 \pmod 8, \text{ if } p \equiv 7 \pmod{16},$$
$$h(-8p) \equiv 2h(-p) \pmod 8, \text{ if } p \equiv 11 \pmod{16},$$

and

$$h(-8p) \equiv -2h(-p) \pmod 8, \text{ if } p \equiv 3 \pmod{16}.$$

Proof. If $p \equiv j \pmod{16}$, $1 \leqq j \leqq 15$, then

(7.8) $$S_{81} \equiv [j/8] \pmod 2.$$

Let $p \equiv 1 \pmod 4$. Then the first two congruences follow from (7.3), (7.8), and Corollary 3.10. Let $p \equiv 3 \pmod 4$. Then the latter four congruences follow from (7.5), (7.8), and the fact that $h(-p)$ is odd.

Corollary 7.5. We have

$$h(-8p) \equiv 0 \pmod 4, \text{ if } p \equiv 1,7 \pmod 8$$

and

$$h(-8p) \equiv 2 \pmod 4, \text{ if } p \equiv 3,5 \pmod 8.$$

Proof. Let $p \equiv 1 \pmod 4$, and suppose that $p \equiv j \pmod{16}$, $1 \leqq j \leqq 15$. Then

(7.9) $$S_{81} - S_{84} \equiv [j/8] - [j/2] + [3j/8] \pmod 2.$$

The congruences for $p \equiv 1 \pmod 4$ follow from (7.3), (7.4), and (7.9).

Let $p \equiv 3 \pmod 4$, and suppose that $p \equiv j \pmod 8$, $1 \leqq j \leqq 7$. Then

(7.10) $$S_{83} - S_{84} \equiv -[j/2] - [j/4] \pmod 2.$$

The congruences for $p \equiv 3 \pmod 4$ follow from (7.6), (7.7), and (7.10).

COROLLARY 7.6. We have

$$h(-40p) \equiv 0 \pmod 8, \text{ if } p \equiv 1, 9, 31, 39 \pmod{40}$$

and

$$h(-40p) \equiv 4 \pmod 8, \text{ if } p \equiv 11, 19, 21, 29 \pmod{40}.$$

Proof. The congruences follow from (5.13) and Corollary 7.5.

The character sums of this section were studied in great detail from an elementary viewpoint by Osborn [50] and Glaisher [27], [28], [29]. Some of the class number formulas in this section can be traced back to Gauss [26] with the proofs given by Dedekind [21]. The formulas

$$(7.11) \qquad \frac{1}{2} h(-8d) = S_{81}(\chi_d) - S_{84}(\chi_d)$$

and

$$(7.12) \qquad \frac{1}{2} h(8d) = S_{82}(\chi_{-d}) + S_{83}(\chi_{-d})$$

are due to Dirichlet [23]. Proofs of (7.11) and (7.12) were also given by Lerch [44, pp. 407, 409]. Pepin [51], Hurwitz [40], Glaisher [29], Holden [39], Karpinski [42], and Rédei [57] have also derived class number formulas in terms of S_{8i}, $1 \leq i \leq 4$.

For $p \equiv 1 \pmod 8$, Corollary 7.5 was first established by Lerch [45, p. 225]. Brown [14] has proven Corollary 7.5 and all the congruences of Corollary 7.4 involving a single class number. He has also pointed out (personal communication) that the remaining congruences of Corollary 7.4 may be deduced from his work [14] and a paper of Hasse [35]. The latter author [32] has also proved Corollary 7.5 for $p \equiv 7 \pmod 8$. As indicated in the Introduction, Corollaries 7.4 and 7.5 have also been proven by Pizer [52]. The special case of Corollary 7.5 when $p \equiv 19 \pmod{24}$ was brought into prominence by Stark [59]. See also [13].

8. SUMS OVER INTERVALS OF LENGTH $k/10$.

As with intervals of length $k/5$, we are able to establish theorems about positive sums for odd χ only.

THEOREM 8.1. Let χ be odd and put $\chi_{5k}(n) = \left(\frac{n}{5}\right) \chi(n)$. Then