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The congruences for  $p \equiv 3 \pmod{4}$  follow from (7.6), (7.7), and (7.10).

COROLLARY 7.6. We have

$$h(-40p) \equiv 0 \pmod{8}, \text{ if } p \equiv 1, 9, 31, 39 \pmod{40}$$

and

$$h(-40p) \equiv 4 \pmod{8}, \text{ if } p \equiv 11, 19, 21, 29 \pmod{40}.$$

*Proof.* The congruences follow from (5.13) and Corollary 7.5.

The character sums of this section were studied in great detail from an elementary viewpoint by Osborn [50] and Glaisher [27], [28], [29]. Some of the class number formulas in this section can be traced back to Gauss [26] with the proofs given by Dedekind [21]. The formulas

$$(7.11) \quad \frac{1}{2} h(-8d) = S_{81}(\chi_d) - S_{84}(\chi_d)$$

and

$$(7.12) \quad \frac{1}{2} h(8d) = S_{82}(\chi_{-d}) + S_{83}(\chi_{-d})$$

are due to Dirichlet [23]. Proofs of (7.11) and (7.12) were also given by Lerch [44, pp. 407, 409]. Pepin [51], Hurwitz [40], Glaisher [29], Holden [39], Karpinski [42], and Rédei [57] have also derived class number formulas in terms of  $S_{8i}$ ,  $1 \leq i \leq 4$ .

For  $p \equiv 1 \pmod{8}$ , Corollary 7.5 was first established by Lerch [45, p. 225]. Brown [14] has proven Corollary 7.5 and all the congruences of Corollary 7.4 involving a single class number. He has also pointed out (personal communication) that the remaining congruences of Corollary 7.4 may be deduced from his work [14] and a paper of Hasse [35]. The latter author [32] has also proved Corollary 7.5 for  $p \equiv 7 \pmod{8}$ . As indicated in the Introduction, Corollaries 7.4 and 7.5 have also been proven by Pizer [52]. The special case of Corollary 7.5 when  $p \equiv 19 \pmod{24}$  was brought into prominence by Stark [59]. See also [13].

## 8. SUMS OVER INTERVALS OF LENGTH $k/10$ .

As with intervals of length  $k/5$ , we are able to establish theorems about positive sums for odd  $\chi$  only.

THEOREM 8.1. Let  $\chi$  be odd and put  $\chi_{5k}(n) = \binom{n}{5} \chi(n)$ . Then

$$S_{10,1} = \frac{G(\chi)}{4\pi i} \{ [4 + \{1 - \bar{\chi}(2)\} \{\bar{\chi}(5) - 1\}] L(1, \bar{\chi}) \\ - 5^{1/2} [1 + \bar{\chi}(2)] L(1, \bar{\chi}_{5k}) \},$$

$$S_{10,2} = \frac{G(\chi)}{4\pi i} \{ [2 - \bar{\chi}(2)] [1 - \bar{\chi}(5)] L(1, \bar{\chi}) \\ + 5^{1/2} \bar{\chi}(2) L(1, \bar{\chi}_{5k}) \},$$

$$S_{10,3} = \frac{G(\chi)}{4\pi i} \{ [2 - \bar{\chi}(2)] [\bar{\chi}(5) - 1] L(1, \bar{\chi}) \\ + 5^{1/2} [2 + \bar{\chi}(2)] L(1, \bar{\chi}_{5k}) \},$$

$$S_{10,4} = \frac{G(\chi)}{4\pi i} \{ [2 - \bar{\chi}(2)] [1 - \bar{\chi}(5)] L(1, \bar{\chi}) \\ - 5^{1/2} \bar{\chi}(2) L(1, \bar{\chi}_{5k}) \},$$

and

$$S_{10,5} = \frac{G(\chi)}{4\pi i} \{ [3 - 4\bar{\chi}(2) + \bar{\chi}(5)] L(1, \bar{\chi}) \\ - 5^{1/2} L(1, \bar{\chi}_{5k}) \}.$$

COROLLARY 8.2. If  $d < 0$ , we have

$$S_{10,1} > 0, \quad \text{if } \chi(2) = -1 \text{ and } \chi(5) \neq -1;$$

$$S_{10,1} = 0, \quad \text{if } \chi(2) = -1 \text{ and } \chi(5) = -1;$$

$$S_{10,2} > 0, \quad \text{if } \chi(2) = 1, \text{ or if } \chi(2) = 0 \text{ and } \chi(5) \neq 1;$$

$$S_{10,2} = 0, \quad \text{if } \chi(2) = 0 \text{ and } \chi(5) = 1;$$

$$S_{10,2} < 0, \quad \text{if } \chi(2) = -1 \text{ and } \chi(5) = 1;$$

$$S_{10,3} > 0, \quad \text{if } \chi(5) = 1;$$

$$S_{10,4} > 0, \quad \text{if } \chi(2) = -1, \text{ or if } \chi(2) = 0 \text{ and } \chi(5) \neq 1;$$

$$S_{10,4} = 0, \quad \text{if } \chi(2) = 0 \text{ and } \chi(5) = 1;$$

$$S_{10,4} < 0, \quad \text{if } \chi(2) = \chi(5) = 1;$$

and

$$S_{10,5} < 0, \quad \text{if } \chi(2) = 1.$$

We shall refrain from writing down any of the class number formulas arising from Theorem 8.1, since no further congruences for class numbers may be deduced. The sums  $S_{10,i}$ ,  $1 \leq i \leq 5$ , appear to have been previously discussed only by Karpinski [42] and by Rédei [57] in connection with class numbers.