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10. Sums over intervals of length $k/16$

Although $S_{16,i}$, $1 \leq i \leq 8$, may be expressed in terms of Dirichlet L-functions at the value 1 by the methods of the previous sections, in each case, L-functions with complex characters arise. Thus, our methods do not enable us to make any conclusions about the sign of $S_{16,i}$, $1 \leq i \leq 8$. However, we are able to prove the following result.

Theorem 10.1. Let χ be even and put $\chi_{4k} = \chi_4 \chi$ and $\chi_{8k} = \chi_4 \chi_8 \chi$. Then

$$S_{16,1} + S_{16,8} = \frac{G(\chi)}{2\pi} \{ \bar{\chi}(4) L(1, \bar{\chi}_{4k}) + 2^{1/2} \bar{\chi}(2) L(1, \bar{\chi}_{8k}) \},$$

$$S_{16,2} + S_{16,7} = \frac{G(\chi)}{2\pi} \{ [2\bar{\chi}(2) - \bar{\chi}(4)] L(1, \bar{\chi}_{4k}) - 2^{1/2} \bar{\chi}(2) L(1, \bar{\chi}_{8k}) \},$$

$$S_{16,3} + S_{16,6} = \frac{G(\chi)}{2\pi} \{ - [2\bar{\chi}(2) + \bar{\chi}(4)] L(1, \bar{\chi}_{4k}) + 2^{1/2} \bar{\chi}(2) L(1, \bar{\chi}_{8k}) \},$$

and

$$S_{16,4} + S_{16,5} = \frac{G(\chi)}{2\pi} \{ \bar{\chi}(4) L(1, \bar{\chi}_{4k}) - 2^{1/2} \bar{\chi}(2) L(1, \bar{\chi}_{8k}) \}.$$

Let χ be odd and put $\chi_{8k} = \chi_8 \chi$. Then

$$S_{16,1} - S_{16,8} = \frac{G(\chi)}{2\pi i} \{ [2\bar{\chi}(2) + \bar{\chi}(4) + \bar{\chi}(8)] [1 - \frac{1}{2} \bar{\chi}(2)] L(1, \bar{\chi}) - 2^{1/2} \bar{\chi}(2) L(1, \bar{\chi}_{8k}) \},$$

$$S_{16,2} - S_{16,7} = \frac{G(\chi)}{2\pi i} \{ [\bar{\chi}(4) - \bar{\chi}(8)] [1 - \frac{1}{2} \bar{\chi}(2)] L(1, \bar{\chi}) + 2^{1/2} \bar{\chi}(2) L(1, \bar{\chi}_{8k}) \},$$

$$S_{16,3} - S_{16,6} = \frac{G(\chi)}{2\pi i} \{ [\bar{\chi}(8) - \bar{\chi}(4)] [1 - \frac{1}{2} \bar{\chi}(2)] L(1, \bar{\chi}) + 2^{1/2} \bar{\chi}(2) L(1, \bar{\chi}_{8k}) \},$$

and

$$S_{16,4} - S_{16,5} = \frac{G(\chi)}{2\pi i} \{ [2\bar{\chi}(2) - \bar{\chi}(4) - \bar{\chi}(8)] [1 - \frac{1}{2} \bar{\chi}(2)] L(1, \bar{\chi})$$
$$- 2^{1/2} \bar{\chi}(2) L(1, \bar{\chi}_{8k}) \} .$$

Corollary 10.2. If d is odd and positive, then

$$S_{16,1} + S_{16,8} > 0 , \quad \text{if} \quad \chi(2) = 1 ,$$

and

$$S_{16,4} + S_{16,5} > 0 , \quad \text{if} \quad \chi(2) = -1 .$$

If d is odd and negative, then

$$S_{16,2} - S_{16,7} > 0 , \quad \text{if} \quad \chi(2) = 1 ,$$
$$S_{16,3} - S_{16,6} > 0 , \quad \text{if} \quad \chi(2) = 1 ,$$
$$S_{16,3} - S_{16,6} < 0 , \quad \text{if} \quad \chi(2) = -1 ,$$

and

$$S_{16,4} - S_{16,5} < 0 , \quad \text{if} \quad \chi(2) = 1 .$$

11. Sums over intervals of length $k/24$.

For intervals of length $k/24$, a complete statement of Theorem 11.1 for both even and odd characters would require 24 formulas. Because of limitations of space, we state just 2 of the formulas for $S_{24,i}(\chi)$, where $1 \leq i \leq 12$ and χ is even or odd.

Theorem 11.1. Let χ be even. Let $\chi_{3k}(n) = \left(\frac{n}{3}\right) \chi(n)$, $\chi_{4k}(n) = \chi_4(n) \chi(n)$, $\chi_{8k}(n) = \chi_4(n) \chi_8(n) \chi(n)$, and $\chi_{24k}(n) = \left(\frac{n}{3}\right) \chi_8(n) \chi(n)$. Then

$$S_{24,1} = \frac{G(\chi)}{2\pi} \left\{ \frac{1}{2} \bar{\chi}(2) [1 + \bar{\chi}(3)] L(1, \bar{\chi}_{4k}) \right.$$
$$+ \frac{1}{4} 3^{1/2} \bar{\chi}(4) [1 + \bar{\chi}(2)] L(1, \bar{\chi}_{3k}) + 2^{-1/2} [\bar{\chi}(3) - 1] L(1, \bar{\chi}_{8k})$$
$$\left. + (3/2)^{1/2} L(1, \bar{\chi}_{24k}) \right\} .$$