COROLLARY 13.14. If $\chi$ is even and real, then $S_{11}(\chi, 2) > 0$ and $S_{21}(\chi, 2) < 0$.

THEOREM 13.15. Let $\chi$ be odd. Then

$$S_{11}(\chi, 2) = \frac{iG(\chi)k^2}{\pi} L(1, \bar{\chi}) \tag{13.1}$$

and

$$S_{21}(\chi, 2) = \frac{iG(\chi)k^2}{\pi} \left\{ \frac{1}{4}[\bar{\chi}(2) - 1] L(1, \bar{\chi}) + \frac{1}{\pi^2}\left[1 - \frac{1}{8}\bar{\chi}(2)\right] L(3, \bar{\chi}) \right\}.$$

COROLLARY 13.16. Let $\chi$ be odd and real. Then in all cases, $S_{11}(\chi, 2) < 0$; if $\chi(2) = 1$, then $S_{21}(\chi, 2) < 0$.

If $\chi$ is real, the class number formula corresponding to (13.1) is due to Cauchy [17]. Pepin [51, p. 205], Lerch [44, p. 395], and Ayoub, Chowla, and Walum [3] have also given proofs of (13.1). Of course, any number of formulas could be proven for $\sum_{a \leq n \leq b} \chi(n) n^r$, where $r$ is a positive integer and $a$ and $b$ are rational multiples of $k$. However we are unable to make any more non trivial deductions about the positivity (or negativity) of such character sums. In this connection, see [3] and [25].

## 14. SOME QUESTIONS AND PROBLEMS

In the foregoing work, in order to determine if $S_{ji}$ is of constant sign for classes of real, primitive characters, we expressed $S_{ji}$ as a linear combination of $L$-functions of real characters evaluated as $s = 1$, and then we inspected the coefficients in this linear combination to determine if all were either non-negative or non-positive. In fact, $S_{ji}$ may always be expressed as a linear combination of $L$-functions evaluated at $s = 1$. However, in the general situation, the $L$-functions are associated with complex characters. When non-real characters arise in the representation of $S_{ji}$, we are unable to say anything about the sign of $S_{ji}$. We have attempted to find all instances when $S_{ji}$ can be expressed in terms of $L$-functions of real characters. It is natural to ask if these cases are the only instances when theorems about the non-negativity or non-positivity of $S_{ji}$ are possible. Results of P. D. T. A. Elliott (written communication) appear to indicate that this, indeed, is the case. For example, he has proved the following result. Consider the set of primes $p$ in any residue class, e.g., $p \equiv 1 \pmod 8$, and the associated characters $\chi_p$ of a given fixed order. Then the values of $\arg L(1, \chi_p)$, as $p$ varies, are everywhere dense modulo $2\pi$.

Let us look at just one example where the admittedly scant, numerical evidence seems to suggest otherwise. Let $\chi(n)$ denote the Legendre symbol modulo $p$, where $p \equiv 1 \pmod 4$. Then $S_{51}$ cannot be expressed in terms of $L$-functions with real characters. However, for $p \equiv 1 \pmod 8$ and $p \leqq 30{,}000$, computations show that $S_{51} > 0$. Sufficient conditions for the positivity of $S_{51}$ are that the two series on the right side of (5.14) are positive. For $p \equiv 1 \pmod 8$, are these two series always positive?

There are a few instances for which we are able to express $S_{ji}$ in terms of $L$-functions of real characters and for which we are unable to deduce any theorems on the sign of $S_{ji}$, but for which numerical computations suggest a constant sign. Again, let $\chi(n) = \left(\frac{n}{p}\right)$. For primes $p$ with $p \equiv 7 \pmod 8$ and $p \leqq 200{,}000$, calculations of Duncan Buell show that $h(-5p) < \left\{5 - \left(\frac{5}{p}\right)\right\} h(-p)$, or, equivalently, by Corollary 5.3, that $S_{51} > 0$. Is this true for all $p$ with $p \equiv 7 \pmod 8$?

There are 7 additional cases for intervals of length $p/24$ in which numerical calculations for $p \leqq 30{,}000$ suggest that $S_{24,i}$ may possibly have a constant sign. For $p \equiv 11 \pmod{24}$, $S_{24,3}, S_{24,11} > 0$; for $p \equiv 17 \pmod{24}$, $S_{24,8}, S_{24,9} < 0$; for $p \equiv 19 \pmod{24}$, $S_{24,6} > 0$; and for $p \equiv 23 \pmod{24}$, $S_{24,2} = -S_{24,12} > 0$. It can be shown that the above inequalities have the following implications, which we very tenuously conjecture hold for all primes in the given residue classes. If $p \equiv 11 \pmod{12}$, then $h(-12p) < 2h(-8p) + h(-24p)$; if $p \equiv 11 \pmod{24}$, then $h(-8p) < 2h(-p) + h(-12p)$; if $p \equiv 17 \pmod{24}$, then $2h(-3p) < 2h(-8p) + h(-24p)$ and $h(-8p) < 2h(-3p) + h(-4p)$; and if $p \equiv 19 \pmod{24}$, then $4h(-p) < h(-12p) + h(-24p)$.

S. Chowla has conjectured that if $p$ is a prime with $p \equiv 3 \pmod 8$, then $S_{21}$ assumes every value that is a positive, odd multiple of 3. He has also conjectured that if $p \equiv 7 \pmod 8$, then $S_{21}$ assumes every positive, odd integral value. In other words, Chowla has conjectured that $h(-p)$ assumes every possible odd value for each of the sets of primes $p$ with $p \equiv 3 \pmod 8$ and $p \equiv 7 \pmod 8$. Samuel Wagstaff has done some calculations to test Chowla's conjectures and similar conjectures of the author. All of the calculational data are for $p \leqq 30{,}000$. For $p \equiv 3 \pmod 8$, the largest value for $S_{21}$ is 297. There are only two omissions, 249 and 291.

For $p \equiv 7 \pmod 8$, the largest value for $S_{21}$ is 259. The smallest value not assumed is 163. There are several other values between 163 and 259 that are not assumed. The calculations also strongly support the following conjectures. $S_{41}$ and $S_{31}$, for $p \equiv 1 \pmod 4$; $S_{52}$, for $p \equiv 3 \pmod 4$; $S_{81}$, for $p \equiv 1 \pmod 8$; $S_{82}$, for $p \equiv 7 \pmod 8$; $-S_{84}$, for $p \equiv 5 \pmod 8$; and $S_{12,2}$, for $p \equiv 7 \pmod 8$ and for $p \equiv 11 \pmod{12}$, each assumes all positive, integral values. We refer the reader to the foregoing work here for the translations of these conjectures into conjectures about class numbers.

## REFERENCES


[1] Apostol, Tom M. Quadratic residues and Bernoulli numbers. *Delta 1* (1968/70) pp. 21-31.

[2] Ayoub, Raymond. *An introduction to the analytic theory of numbers.* American Mathematical Society, Providence, 1963.

[3] ——, S. Chowla and H. Walum. On sums involving quadratic characters. *J. London Math. Soc. 42* (1967), pp. 152-154.

[4] Barrucand, Pierre and Harvey Cohn. Note on primes of type $x^2 + 32y^2$, class number, and residuacity. *J. Reine Angew. Math. 238* (1969), pp. 67-70.

[5] Berger, A. Sur une sommation de quelques séries. *Nova Acta Regiae Soc. Sci. Upsaliensis 12* (1884), 31 pp.

[6] Berndt, Bruce C. Character analogues of the Poisson and Euler-Maclaurin summation formulas with applications. *J. Number Theory 7* (1975), pp. 413-445.

[7] —— Periodic Bernoulli numbers, summation formulas and applications. *Theory and application of special functions.* Richard A. Askey, ed., Academic Press, New York, 1975, pp. 143-189.

[8] —— and S. Chowla. Zero sums of the Legendre symbol. *Nordisk Mat. Tidskr. 22* (1974), pp. 5-8.

[9] —— and Lowell Schoenfeld. Periodic analogues of the Euler-Maclaurin and Poisson summation formulas with applications to number theory. *Acta Arith. 28* (1975), pp. 23-68.

[10] Brown, Ezra. The class number of $Q(\sqrt{-p})$, for $p \equiv 1 \pmod 8$ a prime. *Proc. Amer. Math. Soc. 31* (1972), pp. 381-383.

[11] —— The power of 2 dividing the class-number of a binary quadratic discriminant. *J. Number Theory 5* (1973), pp. 413-419.

[12] —— Class numbers of complex quadratic fields. *J. Number Theory 6* (1974), pp. 185-191.

[13] —— A lemma of Stark. *J. Reine Angew. Math. 265* (1974), p. 26.

[14] —— Class numbers of quadratic fields. *Symp. Mat. 15* (1975), pp. 403-411.

[15] —— and Charles J. Perry. Class numbers of imaginary quadratic fields having exactly three discriminant divisors. *J. Reine Angew. Math. 260* (1973), pp. 31-34.

[16] Carlitz, L. Some sums connected with quadratic residues. *Proc. Amer. Math. Soc. 4* (1953), pp. 12-15.

[17] Cauchy, A. L. *Note XII, Œuvres (1), Tome III.* Gauthier-Villars, Paris, 1882, pp. 359-390.

[18] Chowla, S. On a problem of analytic number theory. *Proc. Nat. Inst. Sci. India 13* (1947), pp. 231-232.