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THE INFINITUDE OF PRIMES; A PROOF USING CONTINUED FRACTIONS

by C. W. Barnes

1. Introduction

Several proofs that there exist infinitely many primes have the elegance of Euclid's classic proof. The various proofs proceed mainly along the lines of Euclid; namely assuming that there are only finitely many primes and constructing an integer greater than one which is not divisible by any of the existing primes.

The proof of Kummer [2] is slightly different. Suppose that the primes are $p_1, p_2, ..., p_t$, $t \geqslant 2$. If we set $n = \prod_{i=1}^{t} p_i$ then 1 is the only integer less than n which is relatively prime to n. However, $1 < n - 1 < n$ and we see that $(n-1, n) = 1$. For if $(n-1, n) = d > 1$ there is a prime p_i such that $p_i \mid d$. Hence $p_i \mid (n-1)$ and $p_i \mid n$. It follows that p_i divides the difference of n and $n - 1$ or $p_i \mid 1$ which cannot hold. Thus there are at least two positive integers less than n and relatively prime to n, a contradiction.

A proof due to Pólya [4] is well-known. It depends on the fact that any two distinct Fermat numbers $F_n = 2^{2^n} + 1$ are relatively prime. Thus each of $F_1, F_2, ..., F_n$ is divisible by an odd prime which does not divide any of the others. Hence it follows that there are at least n odd primes not exceeding F_n for every positive integer n.

Stieltjes [5] gave a proof which may be considered a generalization of that of Euclid. If $p_1, p_2, ..., p_t$ are the existing primes, we write their product in the form mn in any manner. Thus each of $p_1, p_2, ..., p_t$ divides m or n but not both m and n. Therefore $m + n$ is not divisible by any of the existing primes. This is a contradiction since $m + n > 1$ and must be divisible by a prime. If we set $m = 1$ we obtain the proof of Euclid.

A proof given by Braun [1] depends on the same result used by Kummer: if $d \mid a$ and $d \mid b$ for integers a, b, and $d \neq 0$ then $d \mid (ax+by)$ for any integers x and y. Now suppose the existing primes are $p_1, p_2, \ldots, p_t$ with $p_t \geqslant 5$. Write

$$\sum_{i=2}^{t} \frac{1}{p_i} = \frac{m}{n}$$

where

$$m = p_2 p_3 \cdots p_t + p_1 p_3 \cdots p_t + \ldots + p_1 p_2 \cdots p_{t-1}$$

and $n = p_1 p_2 \ldots p_t$. Now $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} > 1$ so that $\frac{m}{n} > 1$. Moreover $m > n$, so that $m > 1$ and thus m has a prime factor p_i. This implies

$$p_i \mid p_1 \cdots p_{i-1} p_{i+1} \cdots p_t$$

and again we have a contradiction.

In the present note we indicate how the theory of simple continued fractions can be used to give a new proof that there exist infinitely many primes. The proof is an application of the theory of periodic continued fractions and the theory of the Pellian equation.

2. Continued Fractions

The necessary material can be found in Perron [3]. We denote the numerators and denominators of the approximants to the simple continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}$$

by A_m and B_m respectively for $m = 0, 1, 2, \ldots$. Thus $A_0 = a_0$, $A_1 = a_0 a_1 + 1$, $B_0 = 1$, $B_0 = a_1$ and for $m \geqslant 1$ we have

$$B_{m+1} = a_{m+1} B_m + B_{m-1} \, . \tag{1}$$

The limit of every infinite periodic simple continued fraction is a quadratic irrational. In particular, if p is a positive integer and

$$x = p + \cfrac{1}{p + \cfrac{1}{p + \ddots}}$$

then we have

(2) $$x = \frac{p + \sqrt{p^2 + 4}}{2}.$$

Suppose d is a positive integer which is not the square of an integer. The Diophantine equation

(3) $$x^2 - dy^2 = -1$$

is often called the non-Pellian equation. If the simple continued fraction for $\sqrt{d}$, which is necessarily periodic, has a period consisting of an odd number, m, of terms, then (3) has a solution. In this case every positive solution is of the form $x = A_i, y = B_i$, for $i = qm - 1$ with q odd.

3. The Basic Result

We use the above results to establish the Theorem. There exist infinitely many primes.

Proof. Assume that there are only finitely many primes $p_1, p_2, ..., p_t$ where $p_1 = 2$. Let $p = \prod_{i=1}^{t} p_i$ and $q = \prod_{i=2}^{t} p_i$ so that q is the product of the odd primes, and hence $q > 1$. Define x by (2). Then in terms of q we have

$$x = q + \sqrt{q^2 + 1}.$$

Since $q^2 + 1 > 1$ and $p_i \nmid (q^2+1)$ for $i = 2, 3, ..., t$ it follows that $q^2 + 1$ is a power of 2 since 2 is the only remaining prime. Moreover, $q^2 + 1$ must be an odd power of 2 since x is irrational. Thus $q^2 + 1 = 2^{2l+1}$ or

$$q^2 - 2(2^l)^2 = -1$$

and it follows that the non-Pellian equation

$$x^2 - 2y^2 = -1$$

have a solution $x = q, y = 2^l$. Hence $\frac{q}{2^l}$ is an even approximant to the continued fraction for $\sqrt{2}$. We have

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}$$

and using (1) we easily verify by induction, for this particular continued fraction, that for every $m > 0$ B_{2m} is an odd integer greater than one. Therefore we must have

$$\frac{q}{2^l} = \frac{A_0}{B_0} = \frac{1}{1}$$

and $q = 1$ since $(q, 2^l) = 1$. This is a contradiction since $q > 1$. The same contradiction follows from $2^l = 1$ since this implies $l = 0$ and thus

$$q^2 + 1 = 2^{2l+1} = 2 .$$

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