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# THE INFINITUDE OF PRIMES; A PROOF USING CONTINUED FRACTIONS

by C. W. BARNES

## 1. INTRODUCTION

Several proofs that there exist infinitely many primes have the elegance of Euclid's classic proof. The various proofs proceed mainly along the lines of Euclid; namely assuming that there are only finitely many primes and constructing an integer greater than one which is not divisible by any of the existing primes.

The proof of Kummer [2] is slightly different. Suppose that the primes are  $p_1, p_2, \dots, p_t$ ,  $t \geq 2$ . If we set  $n = \prod_{i=1}^t p_i$  then 1 is the only integer less than  $n$  which is relatively prime to  $n$ . However,  $1 < n - 1 < n$  and we see that  $(n-1, n) = 1$ . For if  $(n-1, n) = d > 1$  there is a prime  $p_i$  such that  $p_i \mid d$ . Hence  $p_i \mid (n-1)$  and  $p_i \mid n$ . It follows that  $p_i$  divides the difference of  $n$  and  $n - 1$  or  $p_i \mid 1$  which cannot hold. Thus there are at least two positive integers less than  $n$  and relatively prime to  $n$ , a contradiction.

A proof due to Pólya [4] is well-known. It depends on the fact that any two distinct Fermat numbers  $F_n = 2^{2^n} + 1$  are relatively prime. Thus each of  $F_1, F_2, \dots, F_n$  is divisible by an odd prime which does not divide any of the others. Hence it follows that there are at least  $n$  odd primes not exceeding  $F_n$  for every positive integer  $n$ .

Stieltjes [5] gave a proof which may be considered a generalization of that of Euclid. If  $p_1, p_2, \dots, p_t$  are the existing primes, we write their product in the form  $mn$  in any manner. Thus each of  $p_1, p_2, \dots, p_t$  divides  $m$  or  $n$  but not both  $m$  and  $n$ . Therefore  $m + n$  is not divisible by any of the existing primes. This is a contradiction since  $m + n > 1$  and must be divisible by a prime. If we set  $m = 1$  we obtain the proof of Euclid.

A proof given by Braun [1] depends on the same result used by Kummer: if  $d \mid a$  and  $d \mid b$  for integers  $a, b$ , and  $d \neq 0$  then  $d \mid (ax + by)$  for any

integers  $x$  and  $y$ . Now suppose the existing primes are  $p_1, p_2, \dots, p_t$  with  $p_t \geq 5$ . Write

$$\sum_{i=2}^t \frac{1}{p_i} = \frac{m}{n}$$

where

$$m = p_2 p_3 \dots p_t + p_1 p_3 \dots p_t + \dots + p_1 p_2 \dots p_{t-1}$$

and  $n = p_1 p_2 \dots p_t$ . Now  $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} > 1$  so that  $\frac{m}{n} > 1$ . Moreover  $m > n$ , so that  $m > 1$  and thus  $m$  has a prime factor  $p_i$ . This implies

$$p_i \mid p_1 \dots p_{i-1} p_{i+1} \dots p_t$$

and again we have a contradiction.

In the present note we indicate how the theory of simple continued fractions can be used to give a new proof that there exist infinitely many primes. The proof is an application of the theory of periodic continued fractions and the theory of the Pellian equation.

## 2. CONTINUED FRACTIONS

The necessary material can be found in Perron [3]. We denote the numerators and denominators of the approximants to the simple continued fraction

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}}$$

by  $A_m$  and  $B_m$  respectively for  $m = 0, 1, 2, \dots$ . Thus  $A_0 = a_0$ ,  $A_1 = a_0 a_1 + 1$ ,  $B_0 = 1$ ,  $B_1 = a_1$  and for  $m \geq 1$  we have

$$(1) \quad B_{m+1} = a_{m+1} B_m + B_{m-1}.$$

The limit of every infinite periodic simple continued fraction is a quadratic irrational. In particular, if  $p$  is a positive integer and

$$x = p + \frac{1}{p + \frac{1}{p + \dots}}$$

then we have