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integers  $x$  and  $y$ . Now suppose the existing primes are  $p_1, p_2, \dots, p_t$  with  $p_t \geq 5$ . Write

$$\sum_{i=2}^t \frac{1}{p_i} = \frac{m}{n}$$

where

$$m = p_2 p_3 \dots p_t + p_1 p_3 \dots p_t + \dots + p_1 p_2 \dots p_{t-1}$$

and  $n = p_1 p_2 \dots p_t$ . Now  $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} > 1$  so that  $\frac{m}{n} > 1$ . Moreover  $m > n$ , so that  $m > 1$  and thus  $m$  has a prime factor  $p_i$ . This implies

$$p_i \mid p_1 \dots p_{i-1} p_{i+1} \dots p_t$$

and again we have a contradiction.

In the present note we indicate how the theory of simple continued fractions can be used to give a new proof that there exist infinitely many primes. The proof is an application of the theory of periodic continued fractions and the theory of the Pellian equation.

## 2. CONTINUED FRACTIONS

The necessary material can be found in Perron [3]. We denote the numerators and denominators of the approximants to the simple continued fraction

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}}$$

by  $A_m$  and  $B_m$  respectively for  $m = 0, 1, 2, \dots$ . Thus  $A_0 = a_0$ ,  $A_1 = a_0 a_1 + 1$ ,  $B_0 = 1$ ,  $B_1 = a_1$  and for  $m \geq 1$  we have

$$(1) \quad B_{m+1} = a_{m+1} B_m + B_{m-1}.$$

The limit of every infinite periodic simple continued fraction is a quadratic irrational. In particular, if  $p$  is a positive integer and

$$x = p + \frac{1}{p + \frac{1}{p + \dots}}$$

then we have

$$(2) \quad x = \frac{p + \sqrt{p^2 + 4}}{2}.$$

Suppose  $d$  is a positive integer which is not the square of an integer. The Diophantine equation

$$(3) \quad x^2 - dy^2 = -1$$

is often called the non-Pellian equation. If the simple continued fraction for  $\sqrt{d}$ , which is necessarily periodic, has a period consisting of an odd number,  $m$ , of terms, then (3) has a solution. In this case every positive solution is of the form  $x = A_i, y = B_i$ , for  $i = qm - 1$  with  $q$  odd.

### 3. THE BASIC RESULT

We use the above results to establish the THEOREM. There exist infinitely many primes.

*Proof.* Assume that there are only finitely many primes  $p_1, p_2, \dots, p_t$  where  $p_1 = 2$ . Let  $p = \prod_{i=1}^t p_i$  and  $q = \prod_{i=2}^t p_i$  so that  $q$  is the product of the odd primes, and hence  $q > 1$ . Define  $x$  by (2). Then in terms of  $q$  we have

$$x = q + \sqrt{q^2 + 1}.$$

Since  $q^2 + 1 > 1$  and  $p_i \nmid (q^2 + 1)$  for  $i = 2, 3, \dots, t$  it follows that  $q^2 + 1$  is a power of 2 since 2 is the only remaining prime. Moreover,  $q^2 + 1$  must be an odd power of 2 since  $x$  is irrational. Thus  $q^2 + 1 = 2^{2l+1}$  or

$$q^2 - 2(2^l)^2 = -1$$

and it follows that the non-Pellian equation

$$x^2 - 2y^2 = -1$$

have a solution  $x = q, y = 2^l$ . Hence  $\frac{q}{2^l}$  is an even approximant to the continued fraction for  $\sqrt{2}$ . We have

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \dots}}$$

and using (1) we easily verify by induction, for this particular continued