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Artikel: THE INFINITUDE OF PRIMES; A PROOF USING CONTINUED FRACTIONS

Kapitel: 2. Continued Fractions

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integers x and y. Now suppose the existing primes are $p_1, p_2, ..., p_t$ with $p_t \geqslant 5$. Write

$$\sum_{i=2}^{t} \frac{1}{p_i} = \frac{m}{n}$$

where

$$m = p_2 p_3 \cdots p_t + p_1 p_3 \cdots p_t + \ldots + p_1 p_2 \cdots p_{t-1}$$

and $n = p_1 p_2 ... p_t$. Now $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} > 1$ so that $\frac{m}{n} > 1$. Moreover $m > n$, so that $m > 1$ and thus m has a prime factor p_i. This implies

$$p_i \mid p_1 \cdots p_{i-1} \, p_{i+1} \cdots p_t$$

and again we have a contradiction.

In the present note we indicate how the theory of simple continued fractions can be used to give a new proof that there exist infinitely many primes. The proof is an application of the theory of periodic continued fractions and the theory of the Pellian equation.

2. Continued Fractions

The necessary material can be found in Perron [3]. We denote the numerators and denominators of the approximants to the simple continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}$$

by A_m and B_m respectively for $m = 0, 1, 2, ...$. Thus $A_0 = a_0$, $A_1 = a_0 a_1 + 1$, $B_0 = 1$, $B_0 = a_1$ and for $m \geqslant 1$ we have

$$B_{m+1} = a_{m+1} B_m + B_{m-1} \,. \tag{1}$$

The limit of every infinite periodic simple continued fraction is a quadratic irrational. In particular, if p is a positive integer and

$$x = p + \cfrac{1}{p + \cfrac{1}{p + \ddots}}$$

then we have

$$x = \frac{p + \sqrt{p^2 + 4}}{2}. \tag{2}$$

Suppose d is a positive integer which is not the square of an integer. The Diophantine equation

$$x^2 - dy^2 = -1 \tag{3}$$

is often called the non-Pellian equation. If the simple continued fraction for $\sqrt{d}$, which is necessarily periodic, has a period consisting of an odd number, m, of terms, then (3) has a solution. In this case every positive solution is of the form $x = A_i$, $y = B_i$, for $i = qm - 1$ with q odd.

3. The Basic Result

We use the above results to establish the Theorem. There exist infinitely many primes.

Proof. Assume that there are only finitely many primes $p_1, p_2, ..., p_t$ where $p_1 = 2$. Let $p = \prod_{i=1}^{t} p_i$ and $q = \prod_{i=2}^{t} p_i$ so that q is the product of the odd primes, and hence $q > 1$. Define x by (2). Then in terms of q we have

$$x = q + \sqrt{q^2 + 1}.$$

Since $q^2 + 1 > 1$ and $p_i \nmid (q^2+1)$ for $i = 2, 3, ..., t$ it follows that $q^2 + 1$ is a power of 2 since 2 is the only remaining prime. Moreover, $q^2 + 1$ must be an odd power of 2 since x is irrational. Thus $q^2 + 1 = 2^{2l+1}$ or

$$q^2 - 2(2^l)^2 = -1$$

and it follows that the non-Pellian equation

$$x^2 - 2y^2 = -1$$

have a solution $x = q$, $y = 2^l$. Hence $\frac{q}{2^l}$ is an even approximant to the continued fraction for $\sqrt{2}$. We have

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}.$$

and using (1) we easily verify by induction, for this particular continued