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Artikel: THE INFINITUDE OF PRIMES; A PROOF USING CONTINUED FRACTIONS

Kapitel: 3. The Basic Result

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(2) $$x = \frac{p + \sqrt{p^2 + 4}}{2}.$$

Suppose d is a positive integer which is not the square of an integer. The Diophantine equation

(3) $$x^2 - dy^2 = -1$$

is often called the non-Pellian equation. If the simple continued fraction for $\sqrt{d}$, which is necessarily periodic, has a period consisting of an odd number, m, of terms, then (3) has a solution. In this case every positive solution is of the form $x = A_i$, $y = B_i$, for $i = qm - 1$ with q odd.

3. The Basic Result

We use the above results to establish the Theorem. There exist infinitely many primes.

Proof. Assume that there are only finitely many primes $p_1, p_2, \ldots, p_t$ where $p_1 = 2$. Let $p = \prod_{i=1}^{t} p_i$ and $q = \prod_{i=2}^{t} p_i$ so that q is the product of the odd primes, and hence $q > 1$. Define x by (2). Then in terms of q we have

$$x = q + \sqrt{q^2 + 1}.$$

Since $q^2 + 1 > 1$ and $p_i \nmid (q^2+1)$ for $i = 2, 3, \ldots, t$ it follows that $q^2 + 1$ is a power of 2 since 2 is the only remaining prime. Moreover, $q^2 + 1$ must be an odd power of 2 since x is irrational. Thus $q^2 + 1 = 2^{2l+1}$ or

$$q^2 - 2(2^l)^2 = -1$$

and it follows that the non-Pellian equation

$$x^2 - 2y^2 = -1$$

have a solution $x = q$, $y = 2^l$. Hence $\frac{q}{2^l}$ is an even approximant to the continued fraction for $\sqrt{2}$. We have

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}$$

and using (1) we easily verify by induction, for this particular continued fraction, that for every $m > 0$ B_{2m} is an odd integer greater than one. Therefore we must have

$$\frac{q}{2^l} = \frac{A_0}{B_0} = \frac{1}{1}$$

and $q = 1$ since $(q, 2^l) = 1$. This is a contradiction since $q > 1$. The same contradiction follows from $2^l = 1$ since this implies $l = 0$ and thus

$$q^2 + 1 = 2^{2l+1} = 2 .$$

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