

**Zeitschrift:** L'Enseignement Mathématique  
**Band:** 22 (1976)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** THE INFINITUDE OF PRIMES; A PROOF USING CONTINUED FRACTIONS  
**Kapitel:** 3. The Basic Result  
**Autor:** Barnes, C. W.  
**DOI:** <https://doi.org/10.5169/seals-48190>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. [Siehe Rechtliche Hinweise.](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. [Voir Informations légales.](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. [See Legal notice.](#)

**Download PDF:** 15.10.2024

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

$$(2) \quad x = \frac{p + \sqrt{p^2 + 4}}{2}.$$

Suppose  $d$  is a positive integer which is not the square of an integer. The Diophantine equation

$$(3) \quad x^2 - dy^2 = -1$$

is often called the non-Pellian equation. If the simple continued fraction for  $\sqrt{d}$ , which is necessarily periodic, has a period consisting of an odd number,  $m$ , of terms, then (3) has a solution. In this case every positive solution is of the form  $x = A_i, y = B_i$ , for  $i = qm - 1$  with  $q$  odd.

### 3. THE BASIC RESULT

We use the above results to establish the THEOREM. There exist infinitely many primes.

*Proof.* Assume that there are only finitely many primes  $p_1, p_2, \dots, p_t$  where  $p_1 = 2$ . Let  $p = \prod_{i=1}^t p_i$  and  $q = \prod_{i=2}^t p_i$  so that  $q$  is the product of the odd primes, and hence  $q > 1$ . Define  $x$  by (2). Then in terms of  $q$  we have

$$x = q + \sqrt{q^2 + 1}.$$

Since  $q^2 + 1 > 1$  and  $p_i \nmid (q^2 + 1)$  for  $i = 2, 3, \dots, t$  it follows that  $q^2 + 1$  is a power of 2 since 2 is the only remaining prime. Moreover,  $q^2 + 1$  must be an odd power of 2 since  $x$  is irrational. Thus  $q^2 + 1 = 2^{2l+1}$  or

$$q^2 - 2(2^l)^2 = -1$$

and it follows that the non-Pellian equation

$$x^2 - 2y^2 = -1$$

have a solution  $x = q, y = 2^l$ . Hence  $\frac{q}{2^l}$  is an even approximant to the continued fraction for  $\sqrt{2}$ . We have

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \dots}}$$

and using (1) we easily verify by induction, for this particular continued

fraction, that for every  $m > 0$   $B_{2m}$  is an odd integer greater than one. Therefore we must have

$$\frac{q}{2^l} = \frac{A_0}{B_0} = \frac{1}{1}$$

and  $q = 1$  since  $(q, 2^l) = 1$ . This is a contradiction since  $q > 1$ . The same contradiction follows from  $2^l = 1$  since this implies  $l = 0$  and thus

$$q^2 + 1 = 2^{2l+1} = 2.$$

#### REFERENCES

- [1] BRAUN, J. Das Fortschreitungs-gesetz der Primzahlen durch eine transcendente Gleichung exakt dargestellt. *Wiss. Beilage Jahresbericht, Gymm., Trier*, 1899.
- [2] KUMMER, E. *Monatsber. Akad. Wiss. Berlin für 1878, 1879*, pp. 777-8.
- [3] PERRON, Oskar. *Die Lehre von den Kettenbrüchen*. Chelsea, New York, 1951.
- [4] PÓLYA, G. and G. SZEGÖ. *Aufgaben und Lehrsätze aus der Analysis*, vol. 2, pp. 133-142, Dover, New York, 1945.
- [5] STIELTJES, T.J. *Annales fac sc. de Toulouse*, IV (1890).

( Reçu le 3 juillet 1976 )

C.W. Barnes

Department of Mathematics  
University of Mississippi  
Mississippi, 38 677