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THE LIE BRACKET AND THE CURVATURE TENSOR

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1. INTRODUCTION

The purpose of this paper is to present simple, coordinate-free proofs of well-known geometric interpretations (Theorems 1 and 2) of the Lie bracket and curvature tensor (in a C^∞ -manifold with affine connection ∇). These pertain to the traversal of "parallelogram-like" circuits. The standard demonstrations of these interpretations usually make use of finite Taylor expansions in some special coordinate systems (cf. [1, pp. 135-138] for the Lie bracket; [5, pp. 106-108] for the curvature tensor), or repeated application of the multivariable chain rule (cf. [2, pp. 18-19] and [6, pp. 5-38 to 5-42] for the bracket). Spivak ([6, pp. 5-41]) refers to his proof as "an horrendous, but clever, calculation." An application to Lie group theory is given in Corollary 1.

All functions, curves, and vector fields are C^∞ on a C^∞ manifold M. If X is a vector field on M, then an *integral curve* of X is a curve γ (or γ_X) satisfying $\gamma'(t) = X(\gamma(t))$, for all t in domain (γ). If, in addition, $\gamma(0) = p$, we say that γ is an integral curve starting at p. We shall use X_t to denote the *flow* of X, so that $X_t(p) = \gamma(t)$, where γ is an integral curve of X starting at p.

2. THE LIE BRACKET

If f is a function on M, the following is immediate from applying Taylor's Theorem for functions of a real variable to the composition $f \cdot \gamma$, and observing that $(f \cdot \gamma)^{(k)} = X^k f \cdot \gamma$. Throughout this paper, $O(n)$ (n a positive integer) denotes a quantity for which $O(n) / t^n$ is bounded for small t.

LEMMA 1. (Taylor's Theorem for integral curves). If γ is an integral curve of a vector field X and if f is a real-valued function defined in a neighborhood of image (γ), then

$$f(\gamma(t)) - f(\gamma(0)) = \sum_{k=1}^{n} \frac{t^k}{k!} (X^k f)(\gamma(0)) + O(n+1)$$

THEOREM 1. Let X and Y be C^{∞} vector fields on the C^{∞} manifold M. Let $p \in M$ and let σ be the curve difined by

$$\sigma(u) = Y_u X_u Y_{-u} X_{-u} p$$

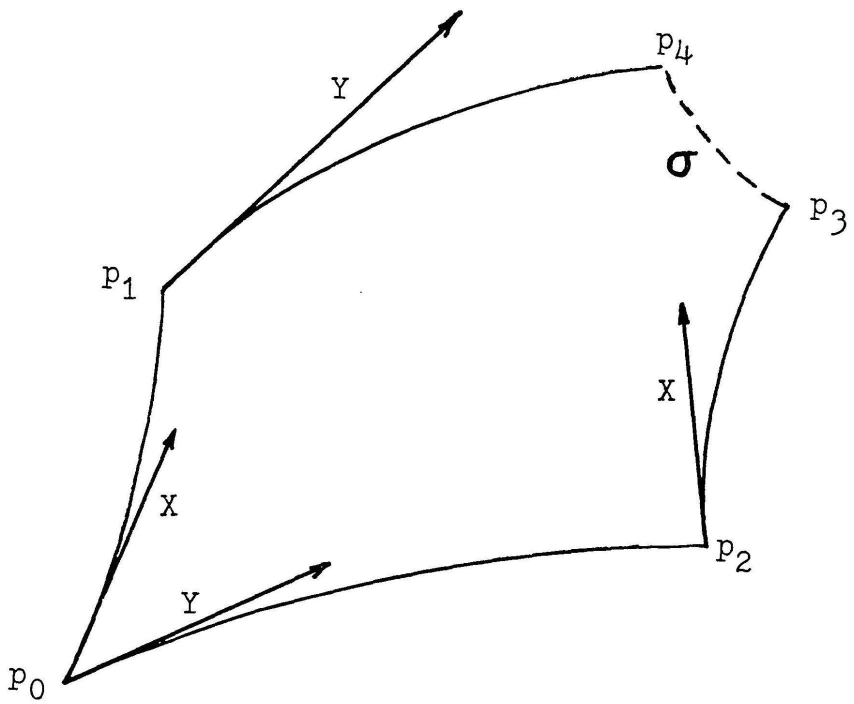
for u sufficiently small. Then for any C^{∞} function f on M,

$$f(\sigma(t)) - f(\sigma(0)) = t^2 [X, Y]_p f + O(3).$$

Accordingly,

$$\lim_{t \to 0} \frac{f(\sigma(\sqrt{t})) - f(\sigma(0))}{t} = [X, Y]_p f$$

and the curve $\beta(u) = \sigma(\sqrt{u})$ satisfies $\beta'(0) = [X, Y]_p$.



Proof: In the figure, the four solid arcs are integral curves of X or Y, as depicted by the arrows, and all are parameterized on the interval $[0, t]$, for t sufficiently small. E.g., $p_2 = \gamma_X(0)$, $p_3 = \gamma_X(t) = X_t(p_2)$, etc. Subscripts denote the point of evaluation: f_i means $f(p_i)$; Xf_i or $X_i f$ means $(Xf)(p_i)$. The point p in the statement of Theorem 1 coincides with p_3 in the figure. We compute $f_4 - f_3$ by applying Lemma 1 to each arc.

(1) $$f_4 - f_1 = tYf_1 + \frac{t^2}{2} Y^2 f_1 + O(3)$$

(2) $$f_1 - f_0 = tXf_0 + \frac{t^2}{2} X^2 f_0 + O(3)$$

(3) $$f_3 - f_2 = tXf_2 + \frac{t^2}{2} X^2 f_2 + O(3)$$

(4) $$f_2 - f_0 = tYf_0 + \frac{t^2}{2} Y^2 f_0 + O(3)$$

Subtracting (3) and (4) from the sum of (1) and (2), and applying Lemma 1 again (up to O (2) only), we obtain

$$f_4 - f_3 = t^2 (XYf - YXf)_0 + \frac{t^3}{2} (XY^2f - YX^2f)_0 + O(3),$$

or

(5) $$f_4 - f_3 = t^2 [X, Y]_0 f + O(3)$$

The meaning of this is that $[X, Y]$ measures the degree to which the circuit $p_3 \to p_2 \to p_0 \to p_1 \to p_4$ fails to be closed. Indeed, if $[X, Y] = 0$, then $p_3 = p_4$ (cf. [1, pp. 134-135]).

If we think of $p = p_3$ as the starting point, and (see figure) define $\sigma(u) = Y_u X_u Y_{-u} X_{-u} p$ (so that $p_4 = \sigma(t)$), we may re-express (5) as

$$f(\sigma(t)) - f(\sigma(0)) = t^2 [X, Y]_0 f + O(3) = t^2 [X, Y]_p f + O(3),$$

since switching to p changes $[X, Y] f$ by an amount which is only of order O (1).

3. A Particular Case

As a special case, assume X and Y are left invariant vector fields on a Lie group G, i.e., elements of $L(G)$, the Lie algebra of G; and take p to be e, the identity element of the group. Since, in this context, $X_t(p) = p \exp(tX)$, for p in G, we have

$$\sigma(t) = \exp(-tX) \exp(-tY) \exp(tX) \exp(tY).$$

If we assume $f(e) = 0$, Theorem 1 yields

$$\begin{aligned} &f(\exp(-tX) \exp(-tY) \exp(tX) \exp(tY)) \\ &= t^2 [X, Y]_e f + O(3) \\ &= f(\exp\{t^2 [X, Y] + O(3)\}) \end{aligned}$$

and so

$$\exp(-tX) \exp(-tY) \exp(tX) \exp(tY) = \exp(t^2 [X, Y] + O(3)).$$