

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 22 (1976)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE LIE BRACKET AND THE CURVATURE TENSOR
Autor: Faber, Richard L.
Kapitel: 2. The Lie Bracket
DOI: <https://doi.org/10.5169/seals-48173>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. [Siehe Rechtliche Hinweise.](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. [Voir Informations légales.](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. [See Legal notice.](#)

Download PDF: 11.01.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

THE LIE BRACKET AND THE CURVATURE TENSOR

by Richard L. FABER

1. INTRODUCTION

The purpose of this paper is to present simple, coordinate-free proofs of well-known geometric interpretations (Theorems 1 and 2) of the Lie bracket and curvature tensor (in a C^∞ -manifold with affine connection ∇). These pertain to the traversal of "parallelogram-like" circuits. The standard demonstrations of these interpretations usually make use of finite Taylor expansions in some special coordinate systems (cf. [1, pp. 135-138] for the Lie bracket; [5, pp. 106-108] for the curvature tensor), or repeated application of the multivariable chain rule (cf. [2, pp. 18-19] and [6, pp. 5-38 to 5-42] for the bracket). Spivak ([6, pp. 5-41]) refers to his proof as "an horrendous, but clever, calculation." An application to Lie group theory is given in Corollary 1.

All functions, curves, and vector fields are C^∞ on a C^∞ manifold M . If X is a vector field on M , then an *integral curve* of X is a curve γ (or γ_X) satisfying $\gamma'(t) = X(\gamma(t))$, for all t in domain (γ) . If, in addition, $\gamma(0) = p$, we say that γ is an integral curve starting at p . We shall use X_t to denote the *flow* of X , so that $X_t(p) = \gamma(t)$, where γ is an integral curve of X starting at p .

2. THE LIE BRACKET

If f is a function on M , the following is immediate from applying Taylor's Theorem for functions of a real variable to the composition $f \cdot \gamma$, and observing that $(f \cdot \gamma)^{(k)} = X^k f \cdot \gamma$. Throughout this paper, $O(n)$ (n a positive integer) denotes a quantity for which $O(n)/t^n$ is bounded for small t .

LEMMA 1. (Taylor's Theorem for integral curves). If γ is an integral curve of a vector field X and if f is a real-valued function defined in a neighborhood of image (γ) , then

$$f(\gamma(t)) - f(\gamma(0)) = \sum_{k=1}^n \frac{t^k}{k!} (X^k f)(\gamma(0)) + O(n+1)$$

THEOREM 1. Let X and Y be C^∞ vector fields on the C^∞ manifold M . Let $p \in M$ and let σ be the curve defined by

$$\sigma(u) = Y_u X_u Y_{-u} X_{-u} p$$

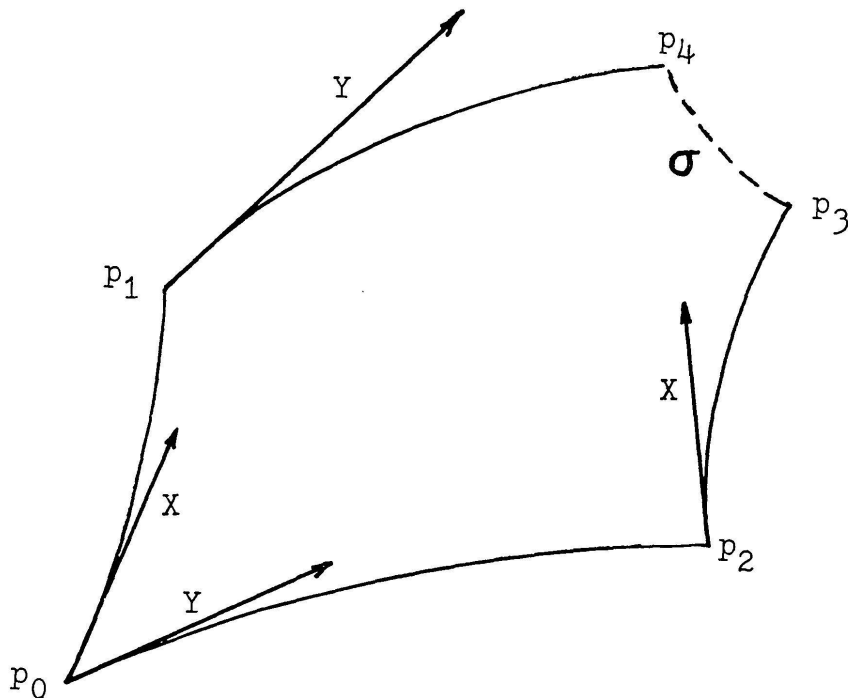
for u sufficiently small. Then for any C^∞ function f on M ,

$$f(\sigma(t)) - f(\sigma(0)) = t^2 [X, Y]_p f + O(3).$$

Accordingly,

$$\lim_{t \rightarrow 0} \frac{f(\sigma(\sqrt{t})) - f(\sigma(0))}{t} = [X, Y]_p f$$

and the curve $\beta(u) = \sigma(\sqrt{u})$ satisfies $\beta'(0) = [X, Y]_p$.



Proof: In the figure, the four solid arcs are integral curves of X or Y , as depicted by the arrows, and all are parameterized on the interval $[0, t]$, for t sufficiently small. E.g., $p_2 = \gamma_X(0)$, $p_3 = \gamma_X(t) = X_t(p_2)$, etc. Subscripts denote the point of evaluation: f_i means $f(p_i)$; Xf_i or $X_i f$ means $(Xf)(p_i)$. The point p in the statement of Theorem 1 coincides with p_3 in the figure. We compute $f_4 - f_3$ by applying Lemma 1 to each arc.

$$(1) \quad f_4 - f_1 = tYf_1 + \frac{t^2}{2} Y^2 f_1 + O(3)$$

$$(2) \quad f_1 - f_0 = tXf_0 + \frac{t^2}{2} X^2 f_0 + O(3)$$

$$(3) \quad f_3 - f_2 = tXf_2 + \frac{t^2}{2} X^2 f_2 + O(3)$$

$$(4) \quad f_2 - f_0 = tYf_0 + \frac{t^2}{2} Y^2 f_0 + O(3)$$

Subtracting (3) and (4) from the sum of (1) and (2), and applying Lemma 1 again (up to $O(2)$ only), we obtain

$$f_4 - f_3 = t^2 (XYf - YXf)_0 + \frac{t^3}{2} (X Y^2 f - Y X^2 f)_0 + O(3),$$

or

$$(5) \quad f_4 - f_3 = t^2 [X, Y]_0 f + O(3)$$

The meaning of this is that $[X, Y]$ measures the degree to which the circuit $p_3 \rightarrow p_2 \rightarrow p_0 \rightarrow p_1 \rightarrow p_4$ fails to be closed. Indeed, if $[X, Y] = 0$, then $p_3 = p_4$ (cf. [1, pp. 134-135]).

If we think of $p = p_3$ as the starting point, and (see figure) define $\sigma(u) = Y_u X_u Y_{-u} X_{-u} p$ (so that $p_4 = \sigma(t)$), we may re-express (5) as

$$f(\sigma(t)) - f(\sigma(0)) = t^2 [X, Y]_0 f + O(3) = t^2 [X, Y]_p f + O(3),$$

since switching to p changes $[X, Y]f$ by an amount which is only of order $O(1)$.

3. A PARTICULAR CASE

As a special case, assume X and Y are left invariant vector fields on a Lie group G , i.e., elements of $L(G)$, the Lie algebra of G ; and take p to be e , the identity element of the group. Since, in this context, $X_t(p) = p \exp(tX)$, for p in G , we have

$$\sigma(t) = \exp(-tX) \exp(-tY) \exp(tX) \exp(tY).$$

If we assume $f(e) = 0$, Theorem 1 yields

$$\begin{aligned} & f(\exp(-tX) \exp(-tY) \exp(tX) \exp(tY)) \\ &= t^2 [X, Y]_e f + O(3) \\ &= f(\exp\{t^2 [X, Y] + O(3)\}) \end{aligned}$$

and so

$$\exp(-tX) \exp(-tY) \exp(tX) \exp(tY) = \exp(t^2 [X, Y] + O(3)).$$