**Zeitschrift:** L'Enseignement Mathématique

**Herausgeber:** Commission Internationale de l'Enseignement Mathématique

**Band:** 22 (1976)

**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** THE LIE BRACKET AND THE CURVATURE TENSOR

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**Kapitel:** 4. The Curvature Tensor

**DOI:** https://doi.org/10.5169/seals-48173

This formula is involved in proving that if $H$ is (algebraically) a subgroup of a Lie group $G$ and if $H$ is a closed subset of $G$, then $H$ is a topological Lie subgroup of $G$ ([3, pp. 96, 105]). Specifically, it implies that $\{ V$ in $L(G) \mid \exp(tV)$ is in $H$, for all $t$ real$\}$ is closed under the bracket. The formula also provides the following geometric interpretation of the bracket $[X, Y]$ on the Lie algebra $L(G)$ of a Lie group $G$.

COROLLARY 1. If $X$ and $Y$ belong to $L(G)$, then the curve

$$t \to \exp(-\sqrt{t}X) \exp(-\sqrt{t}Y) \exp(\sqrt{t}X) \exp(\sqrt{t}Y)$$

has velocity vector $[X, Y]$ at $t = 0$.

## 4. THE CURVATURE TENSOR

Now assume $M$ is furnished with an affine connection (covariant differentiation operator) $\nabla$.

The *curvature tensor* $R$ on $M$ is the $\binom{1}{3}$-tensor (equivalently, the linear-transformation-valued bilinear mapping) $R$ defined by

$$\begin{aligned} R(X, Y)A &= \nabla_X \nabla_Y A - \nabla_Y \nabla_X A - \nabla_{[X,Y]} A \\ &= ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]}) A, \end{aligned}$$

for $X$, $Y$, and $A$ vector fields on $M$. The relationship between this tensor and the Riemann curvature (in a Riemannian manifold) may be found in [4, pp. 72-73], [2, Chapter 9], and [5, pp. 125-127]. Here we shall show its relationship to parallel translation.

Consider the figure again, and let $A$ be any vector field on $M$. We shall compare parallel translation along $p_0 \to p_1 \to p_4$ with that along $p_0 \to p_2 \to p_3$. Then, by adding the curve $\sigma(u) = Y_u X_u Y_{-u} X_{-u} p_3$ defined previously (the dotted curve in the figure), we obtain a closed circuit. We shall need the following.

LEMMA 2. (Taylor's Theorem for parallel translation). Let $X$ be a vector field defined in a neighborhood of a curve $\gamma$, let $T = \gamma'(0)$, and for any $t$ in domain $(\gamma)$, let $\tau_t$ denote parallel translation along $\gamma$ to $\gamma(t)$. Then

$$\tau_0 X(\gamma(t)) - X(\gamma(0)) = \sum_{k=1}^{n} \frac{t^k}{k!} \nabla_T{}^k X + O(n+1).$$

*Proof.* Apply the real-variable Taylor's Theorem to the function $f(t) = \tau_0 X(\gamma(t))$ which has values in a finite dimensional vector space.

$$f'(t) = \lim_{h \to 0} \frac{\tau_0 X(\gamma(t+h)) - \tau_0 X(\gamma(t))}{h}$$

$$= \tau_0 \lim_{h \to 0} \frac{\tau_t X(\gamma(t+h)) - X(\gamma(t))}{h} = \tau_0 \nabla_{\gamma'(t)} X .$$

Inductively, $f^{(n)}(t) = \tau_0 (\nabla_{\gamma'(t)}{}^n X)$ and $f^{(n)}(0) = \nabla_T{}^n X$.

THEOREM 2. Let $X$, $Y$, and $A$ be $C^\infty$ vector fields on the $C^\infty$ manifold $M$ with affine connection $\nabla$. Let $p$ belong to $M$ and consider parallel translation of $A_p$ around the closed circuit consisting of (in order) the integral curves of $-X$, $-Y$, $X$, and $Y$ (each parameterized on $[0, t]$, $t$ small), and (backwards along) the curve $\sigma(u) = Y_u X_u Y_{-u} X_{-u} p$, $0 \leqslant u \leqslant t$ (see figure). If $\Delta A$ is the change in $A_p$ produced by parallel translation around this circuit, then

$$\Delta A = t^2 R(Y, X) A_p + O(3)$$

and hence

$$\lim_{t \to 0} \frac{\Delta A}{t^2} = R(Y, X) A_p .$$

*Proof.* The calculation is similar to that for the Lie bracket in Theorem 1, except that we must use parallel translation to compare vectors at different points. $\tau_i$ denotes parallel translation to $p_i$ along the arc to $p_i$ from the location of the tangent vector in question. Elsewhere, subscripts denote point of evaluation, as before. From Lemma 2, we have

$$(6) \qquad \tau_1 A_4 - A_1 = t \nabla_Y A_1 + \frac{t^2}{2} \nabla_Y{}^2 A_1 + O(3)$$

$$(7) \qquad \tau_0 A_1 - A_0 = t \nabla_X A_0 + \frac{t^2}{2} \nabla_X{}^2 A_0 + O(3)$$

$$(8) \qquad \tau_2 A_3 - A_2 = t \nabla_X A_2 + \frac{t^2}{2} \nabla_X{}^2 A_2 + O(3)$$

$$(9) \qquad \tau_0 A_2 - A_0 = t \nabla_Y A_0 + \frac{t^2}{2} \nabla_Y{}^2 A_0 + O(3)$$

Apply $\tau_0$ to both sides of (6) and (8), obtaining (6′) and (8′), respectively. Subtracting (8′) and (9) from the sum of (6′) and (7), we obtain (via Lemma 2),

$$(10) \qquad \tau_0 \tau_1 A_4 - \tau_0 \tau_2 A_3 = t^2 \left[\nabla_X, \nabla_Y\right] A_0 + O(3)$$

As before, let $\beta(u) = \sigma(\sqrt{u})$, $0 \leqslant u \leqslant t^2$. Using $\beta'(0) = [X, Y]_3$ (from Theorem 1), we may, as in the proof of Lemma 2, show that

$$(11) \qquad \tau_3 A_4 - A_3 = t^2 \nabla_{[X,Y]} A_3 + O(4).$$

Now apply $\tau_4$ to (11) and $\tau_4 \tau_1$ to (10). Taking the difference of the resulting equations and then applying $\tau_3$ to both sides, we obtain

$$\begin{aligned} \Delta A &= \tau_3 \tau_4 \tau_1 \tau_0 \tau_2 A_3 - A_3 \\ &= t^2 \left(\tau_3 \tau_4 \nabla_{[X,Y]} A_3 - \tau_3 \tau_4 \tau_1 \left[\nabla_X, \nabla_Y\right] A_0\right) + O(3) \\ &= t^2 \left(\nabla_{[X,Y]} - \left[\nabla_X, \nabla_Y\right]\right) A_3 + O(3) = - t^2 R(X, Y) A_p + O(3), \end{aligned}$$

since the change produced by dropping the $\tau$'s and switching to $p_3$ may be absorbed in $O(3)$. Thus the theorem follows since $-R(X, Y) = R(Y, X)$.

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*(Reçu le 26 juin 1975)*

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