§1. Siegel's Formula

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out the analogous formula for $\zeta_K(1-2m)$, where $m \geqslant 3$, we find that there really is a difference of this order between the Fourier coefficient we are trying to evaluate and the value of the singular series. The calculation of the singular series is carried out in Section 4.

Finally, in §5 we give conjectures concerning the Fourier coefficients of a certain modular form of weight $4m$ related to the value of $\zeta_K(1-2m)$.

§1. Siegel's Formula

In this section, we will state the formula of Siegel for the value of $\zeta_K(b)$ where K is a totally real algebraic number field and b a negative odd integer. We will also give a brief description of the proof.

We begin by reviewing the main properties of the zeta-function of a field. Let K be an algebraic number field of degree n, and $\mathcal{O}$ the ring of integers in K. For any non-zero ideal $\mathfrak{A}$ of $\mathcal{O}$, the *norm* $N(\mathfrak{A})$ is defined as the number of elements in the quotient $\mathcal{O}/\mathfrak{A}$. For $m = 1, 2, \ldots$, let $i(m)$ denote the number of ideals of $\mathcal{O}$ with norm m. This number is finite for each m and has polynomial growth as $m \to \infty$, and so the series $\Sigma_{m=1}^{\infty} i(m) m^{-s}$ makes sense and is convergent if s is a complex number with sufficiently large real part. The function it defines can be extended meromorphically to the whole s-plane, and the function obtained is denoted $\zeta_K(s)$. Thus we have the two representations.

$$\zeta_K(s) = \sum_{\mathfrak{A} \subset \mathcal{O}} \frac{1}{N(\mathfrak{A})^s} \tag{1}$$

$$= \prod_{\mathfrak{P}} (1 - N(\mathfrak{P})^{-1})^{-s}, \tag{2}$$

provided that $Re(s)$ is large enough. The sum in (1) is to be taken over all non-zero ideals of $\mathcal{O}$, and the product in (2) *(Euler product)* over all prime ideals. The function obtained by analytic continuation has a simple pole at $s = 1$ and is holomorphic everywhere else.

Moreover, the function ζ_K satisfies a *functional equation* relating $\zeta_K(s)$ and $\zeta_K(1-s)$. In the case of a totally real field K (i.e. $K = \mathbf{Q}(\alpha)$ where α satisfies a polynomial of degree n with n real roots), this takes the form

$$F(s) = F(1-s), \tag{3}$$

where

$$F(s) = D^{s/2} \pi^{-ns/2} \Gamma\left(\frac{s}{2}\right)^n \zeta_K(s). \tag{4}$$

(Here D is the discriminant of K.) In particular, we have

$$\zeta_K(-2m) = 0, \tag{5}$$

$$\zeta_K(1-2m) = \{(-1)^m(2m-1)!/2^{2m-1}\pi^{2m}\}^n D^{2m-1/2}\zeta_K(2m)$$
$$(m=1,2,\ldots) \tag{6}$$

It is thus equivalent to give the values of $\zeta_K(s)$ at $s = 2, 4, 6, \ldots$ or at $s = -1, -3, -5, \ldots$; we shall prefer writing our formula for the latter values since, as it turns out, they are always rational numbers. For instance, if $K = \mathbf{Q}$ is the field of rational numbers, then $n = 1$, $D = 1$, $\mathcal{O} = \mathbf{Z}$, and the only ideals are (r) with $r = 1, 2, \ldots$, so

$$\zeta_K(s) = \zeta_{\mathbf{Q}}(s) = \zeta(s) = \sum_{r=1}^{\infty} \frac{1}{r^s} \tag{7}$$

is the ordinary Riemann zeta-function; in this case (6) says

$$\zeta(1-2m) = \frac{(-1)^m(2m-1)!}{2^{2m-1}\pi^{2m}}\zeta(2m) \tag{8}$$

$$= -B_{2m}/2m, \tag{9}$$

where B_i is the i-th Bernoulli number ($B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30, \ldots$) and is always rational.

We now proceed to describe Siegel's formula. We first need some preliminary notation. Recall the definition of the *different* $\mathfrak{d}$ of K: $\mathfrak{d}$ is the inverse of the fractional ideal

$$\mathfrak{d}^{-1} = \{x \in K \mid tr(xy) \in \mathbf{Z}\,(\forall y \in \mathcal{O})\} \tag{10}$$

(here $tr(z) = z^{(1)} + \ldots + z^{(n)}$ denotes the trace of $z \in K$). The ideal $\mathfrak{d}$ is integral, and its norm is related to the discriminant D of K by

$$D = N(\mathfrak{d}). \tag{11}$$

Next, for $r = 0, 1, 2, \ldots$ we define

$$\sigma_r(n) = \sum_{d|n} d^r \quad (n=1,2,3,\ldots) \tag{12}$$

to be the sum of the r-th powers of the positive divisors of n. (This is standard notation.) We generalize this definition to number fields by setting

$$\sigma_r(\mathfrak{A}) = \sum_{\mathfrak{B}|\mathfrak{A}} N(\mathfrak{B})^r \quad (\mathfrak{A} \subset \mathcal{O} \text{ an ideal}). \tag{13}$$

Here the sum is over all ideals $\mathfrak{B}$ of $\mathcal{O}$ which divide (i.e. contain) $\mathfrak{A}$. If $K = \mathbf{Q}$, $\mathcal{O} = \mathbf{Z}$, $\mathfrak{A} = (n)$, this agrees with (12).

Finally, for $l, m = 1, 2, \ldots$, we define

$$s_l^K(2m) = \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0 \\ tr(\nu) = l}} \sigma_{2m-1}\big((\nu)\,\mathfrak{d}\big)\,. \tag{14}$$

The sum extends over all totally positive (i.e. all conjugates positive) elements of the fractional ideal (10) with given trace l (there are only finitely many such elements). Such a ν need not be integral, but the product of the principal ideal (ν) with the different $\mathfrak{d}$ will be an integral ideal, and therefore $\sigma_{2m-1}\big((\nu)\,\mathfrak{d}\big)$ is defined.

We can now state Siegel's formula.

THEOREM (Siegel [9]). *Let* $m = 1, 2, \ldots$ *be a natural number,* K *a totally real algebraic number field,* $n = [K:\mathbf{Q}]$, *and* $h = 2mn$. *Then*

$$\zeta_K(1-2m) = 2^n \sum_{l=1}^{r} b_l(h)\, s_l^K(2m)\,. \tag{15}$$

The numbers $r \geqslant 1$ *and* $b_1(h), \ldots, b_r(h) \in \mathbf{Q}$ *depend only on* h. *In* par-*ticular,*

$$r = \dim_{\mathbf{C}} \mathfrak{M}_h\,, \tag{16}$$

where $\mathfrak{M}_h$ *is the space of modular forms of weight* h; *thus by a well-known formula*

$$r = \begin{cases} [h/12] & \text{if} \quad h \equiv 2 \pmod{12}\,, \\ [h/12] + 1 & \text{if} \quad h \not\equiv 2 \pmod{12}\,, \end{cases} \tag{17}$$

where $[x]$ *denotes the greatest integer* $\leqslant x$.

(We have given a table of the coefficients $b_l(h)$ on page 60, if for no other reason than to emphasize that they really only depend on the integer h and not on the field. The values for h even, $4 \leqslant h \leqslant 24$, were taken from Siegel [9]; the values for $4 \mid h \leqslant 40$ were calculated on the System 370 computer at Bonn.)

Proof of theorem (sketch): Recall that one can define a modular form of weight $2m$ by the *Eisenstein series*

$$G_{2m}(z) = \sum_{\substack{\lambda, \mu \in \mathbf{Z} \\ (\lambda, \mu) \neq (0,0)}} \frac{1}{(\lambda z + \mu)^{2m}} \tag{18}$$

TABLE 1.

The Siegel coefficients $b_l(h)$

h	$b_1(h)$	$b_2(h)$	$b_3(h)$	$b_4(h)$
4	$\frac{1}{240}$			
6	$\frac{-1}{504}$			
8	$\frac{1}{480}$			
10	$\frac{-1}{264}$			
12	$\frac{-1}{8190}$	$\frac{1}{196560}$		
14	$\frac{-1}{24}$			
16	$\frac{-1}{680}$	$\frac{1}{146880}$		
18	$\frac{-22}{3591}$	$\frac{-1}{86184}$		
20	$\frac{-19}{1650}$	$\frac{1}{39600}$		
22	$\frac{-4}{207}$	$\frac{-1}{14904}$		
24	$\frac{-1087}{291200}$	$\frac{1}{1092000}$	$\frac{1}{52416000}$	
28	$\frac{-2529}{259840}$	$\frac{-1}{81200}$	$\frac{1}{15590400}$	
32	$\frac{837}{43520}$	$\frac{-9}{54400}$	$\frac{1}{2611200}$	
36	$\frac{-274486}{29895075}$	$\frac{-899}{28787850}$	$\frac{1}{86363550}$	$\frac{1}{6218175600}$
40	$\frac{-602849}{39067875}$	$\frac{-1773}{14206500}$	$\frac{-1}{7441500}$	$\frac{1}{1250172000}$

($z \in \mathfrak{H}$ = upper half-plane, i.e. $z \in \mathbf{C}$ and $Im(z) > 0$). Since $G_{2m}(z)$ has period 1, it has a Fourier expansion as a power series in $q = e^{2\pi i z}$,

$$G_{2m}(z) \sim a_0 + a_1 q + a_2 q^2 + \dots \tag{19}$$

valid as $z \to i\,\infty$ (i.e. $q \to 0$). Then clearly

$$a_0 = \sum_{\substack{\mu \in \mathbf{Z} \\ \mu \neq 0}} \mu^{-2m} = 2\zeta(2m), \tag{20}$$

and an easy calculation gives

$$a_n = 2 \frac{(2\pi i)^{2m}}{(2m-1)!} \sigma_{2m-1}(n) \quad (n = 1, 2, \dots). \tag{21}$$

In an entirely analogous way, for the field K one can construct a modular form of weight $2m$ in n variables $z_1, \dots, z_n \in \mathfrak{H}$ (the *Hecke-Eisenstein series*) and calculate its Fourier coefficients. By setting $z_1 = \dots = z_n = z$, we obtain a modular form $G_{2m}^K(z)$ in one variable, of weight $2mn = h$, with a known Fourier expansion, namely

$$G_{2m}^K(z) \sim a_0 + a_1 q + a_2 q^2 + \dots \tag{22}$$

with

$$a_0 = \zeta_K(2m), \tag{23}$$

$$a_l = \{(2\pi i)^{2m}/(2m-1)!\}^n D^{-2m+1/2} s_l^K(2m) \quad (l = 1, 2, \dots). \tag{24}$$

On the other hand, since the space $\mathfrak{M}_h$ of modular forms of weight h has finite dimension r, there must be a linear relation among the first $r + 1$ coefficients in the Fourier expansion of any such form, i.e. there must exist numbers $c_{h,0}, c_{h,1}, \dots, c_{h,r}$ depending only on h such that

$$f \in \mathfrak{M}_h, \; f \sim a_0 + a_1 q + a_2 q^2 + \dots$$

$$\Rightarrow c_{h,0} a_0 + c_{h,1} a_1 + \dots + c_{h,r} a_r = 0. \tag{25}$$

Siegel then shows that $c_{h,0}$ is non-zero for all h, so we can set

$$b_l(h) = -c_{h,l}/c_{h,0} \quad (l + 1, \dots, r) \tag{26}$$

to obtain from (25) the relation

$$a_0 = \sum_{l=1}^{r} b_l(h)\, a_l \tag{27}$$

expressing the constant term of a modular form of given weight as a linear combination of finitely many of the other coefficients of its Fourier expansion. Substituting (23) and (24) into (27) gives

$$\zeta_K(2m) = \{(2\pi i)^{2m}/(2m-1)!\}^n D^{-2m+1/2} \sum_{l=1}^{r} b_l(h)\, s_l^K(2m), \tag{28}$$

which in view of the functional equation (6) is equivalent to the assertion of the theorem.

Since the numbers $\sigma_r(\mathfrak{A})$ and hence $s_l^K(2m)$ are clearly (rational) integers, we deduce from (15) not only that $\zeta_K(1-2m)$ is rational, but also that its denominator is bounded by a number depending only on h, i.e. only on the number $1 - 2m$ and the degree of the field K.

We now juggle the terms in the Siegel formula somewhat to rewrite it in a suggestive form. If we substitute the definitions (14) and (13) into equation (15) and invert the order of summation, we obtain

$$\begin{aligned}\zeta_K(1-2m) &= 2^n \sum_{l=1}^{r} b_l(h) \sum_{\substack{\nu\in\mathfrak{d}^{-1}\\ \nu\gg 0\\ tr(\nu)=l}} \sum_{\mathfrak{B}\mid(\nu)\mathfrak{d}} N(\mathfrak{B})^{2m-1}\\ &= \sum_{\mathfrak{B}} w(\mathfrak{B})\, N(\mathfrak{B})^{2m-1},\end{aligned} \tag{29}$$

where the sum is over all non-zero integral ideals $\mathfrak{B}$ and the "weight" $w(\mathfrak{B})$ is defined by

$$w(\mathfrak{B}) = 2^n \sum_{\substack{\nu\in\mathfrak{B}\mathfrak{d}^{-1}\\ \nu\gg 0}} b_{tr(\nu)}(h). \tag{30}$$

The sum in (30) is always finite and is empty for all but finitely many ideals $\mathfrak{B}$ (because $b_l(h) = 0$ for $l > r$) so the sum (29) is in fact finite. Equation (29) is a rather amusing formulation of Siegel's theorem, for if we had just mechanically substituted $s = 1 - 2m$ into (1) without regard for convergence, we would have obtained

$$\zeta_K(1-2m) = \sum_{\mathfrak{B}} N(\mathfrak{B})^{2m-1}, \tag{31}$$

which is of course nonsense, but then equation (29) tells us that it is all right after all, if we just insert "fudge factors" $w(\mathfrak{B})$ to weight the summands: thus one really *can* evaluate $\zeta_K(1-2m)$ by adding up $(2m-1)$-th powers of norms of ideals.

In this connection, it is perhaps worthwhile to observe that the weights $w(\mathfrak{B})$ are not unique. Indeed, given h, we can choose any $r' \geqslant r$ and find coefficients $b_1'(h), \ldots, b_{r'}'(h)$ expressing the constant term of any form $f \in \mathfrak{M}_h$ in terms of the next r' coefficients (such collections b' will form an affine space of dimension $r' - r$). Then Siegel's theorem is valid with the

b_l' in place of the b_l, and similarly using the b_l' in (30) would give other weights making (29) hold.

Finally, for completeness' sake we should mention that Siegel gave a somewhat more general formula than the one stated. If A denotes any ideal class of the field K, then restricting the ideals A in the sum (1) to ideals in the class A gives rise to another meromorphic function, denoted $\zeta(s, A)$. This function also takes on rational values at negative odd integers, and Siegel's formula for these rational numbers is identical to (15) except that one must modify the definition of $\sigma_r(\mathfrak{A})$ by only allowing those ideal divisors $\mathfrak{B}$ in (13) that lie in the class A. In the formulation of Siegel's result just given, this can be simply stated

$$\zeta(1-2m, A) = \sum_{\mathfrak{B} \in A} w(\mathfrak{B}) N(\mathfrak{B})^{2m-1}, \qquad (32)$$

with the same weights $w(\mathfrak{B})$ as before.

§2. Zeta-functions of Quadratic Fields

We now specialize to quadratic fields. A totally real quadratic field can be written uniquely as $\mathbf{Q}(d^{1/2})$ with $d > 1$ a square-free integer. Then it is easy to check that

$$\begin{aligned} D &= d \quad \text{if} \quad d \equiv 1 \pmod 4, \\ D &= 4d \quad \text{if} \quad d \equiv 2 \text{ or } 3 \pmod 4, \end{aligned} \qquad (1)$$

and

$$\mathfrak{d} = (\sqrt{D}), \qquad (2)$$

i.e. the different is a principal ideal. The decomposition of rational primes in the ring of integers $\mathcal{O}$ is described in terms of the primitive character $\chi \pmod D$ defined by

$$\chi(x) = \left(\frac{D}{x}\right) \qquad (3)$$

(here χ is completely multiplicative, and given on primes by: $\chi(p) = 0$ if $p \mid D$; for $p \nmid 2D$, $\chi(p)$ is ± 1 according as D is or is not a quadratic residue (mod p); for $p = 2$ and $D = d$ odd, $\chi(2) = (-1)^{(d-1)/4}$) as follows: if $p = 2, 3, 5, \ldots$ is a rational prime, then the ideal $(p) \subset \mathcal{O}$ decomposes into prime ideals according to the value of $\chi(p)$ —