§3. The Siegel Formula for Quadratic Fields

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while for $K = \mathbf{Q}(\sqrt{13})$

$$\zeta_K(-1)$$

$$= \frac{1}{24 \times 13}\left[1^2 - 2^2 + 3^2 + 4^2 - 5^2 - 6^2 - 7^2 - 8^2 + 9^2 + 10^2 - 11^2 + 12^2\right]$$

$$= \frac{1}{6}. \tag{18}$$

For a more complete discussion of the formulas treated in this section, see Siegel [8].

§3. The Siegel Formula for Quadratic Fields

In this section we shall exploit the simple arithmetic of quadratic fields to evaluate in elementary form the various terms entering into Siegel's formula, thus arriving at an expression for $\zeta_K(1-2m)$ which is elementary in the sense that it involves only rational integers and not algebraic numbers or ideals.

We have to evaluate $s_l^K(2m)$, and to do so we must first know how to compute $\sigma_r(\mathfrak{A})$ for any ideal $\mathfrak{A}$.

Lemma. *Let* $\mathfrak{A}$ *be any ideal of the ring of integers* $\mathcal{O}$ *of a quadratic field* K. *Let* D *be the discriminant of* K *and* $\chi(j) = \left(\frac{D}{j}\right)$ *the associated character (as in* §2). *Then, for any* $r \geqslant 0$,

$$\sigma_r(\mathfrak{A}) = \sum_{j \mid \mathfrak{A}} \chi(j)\, j^r \sigma_r(N/j^2), \tag{1}$$

where $N = N(\mathfrak{A})$ *is the norm of* $\mathfrak{A}$, *the function* σ_r *on the right-hand side is the arithmetic function of* 1 (12), *and the sum is over all positive integers* j *dividing* $\mathfrak{A}$ (i.e. $\nu/j \in \mathcal{O}$ for every $\nu \in \mathfrak{A}$; clearly this implies $j^2 \mid N$, so equation (1) makes sense).

Proof: It is very easy to check that both sides of (1) are multiplicative functions, i.e. $\sigma_r(\mathfrak{A}\mathfrak{B}) = \sigma_r(\mathfrak{A})\,\sigma_r(\mathfrak{B})$ for relatively prime ideals $\mathfrak{A}$ and $\mathfrak{B}$, and similarly for the expression on the right-hand side of (1). It therefore suffices to take $\mathfrak{A}$ to be a power $\mathfrak{P}^m$ of a prime ideal $\mathfrak{P}$. Write $N(\mathfrak{P}) = p^i$

where p is a rational prime and $i = 1$ or 2. Then we can evaluate the left-hand side of (1):

$$\sigma_r(\mathfrak{A}) = \sigma_r(\mathfrak{P}^m) = \sum_{\mathfrak{B} \mid \mathfrak{P}^m} N(\mathfrak{B})^r$$

$$= \sum_{n=0}^{m} N(\mathfrak{P}^n)^r = \sum_{n=0}^{m} p^{inr} = \sigma_{ir}(p^m). \qquad (2)$$

To evaluate the right-hand side of (1), we must distinguish three cases, according to the value of $\chi(p)$.

Case 1. $\chi(p) = 1$, $(p) = \mathfrak{P}\mathfrak{P}'$ ($\mathfrak{P}'$ = conjugate of $\mathfrak{P}$). Then $N(\mathfrak{A}) = N(\mathfrak{P})^m = p^m$. Clearly $j \mid \mathfrak{A} \Rightarrow j = 1$, for j can only be a power of p (since $j \mid N(\mathfrak{A})$) and cannot be divisible by p (because $\mathfrak{P}' \mid p$, $\mathfrak{P}' \nmid \mathfrak{A}$). Hence the sum in (1) has only one term $\sigma_r(N) = \sigma_r(p^m)$, in agreement with (2).

Case 2. $\chi(p) = 0$, $(p) = \mathfrak{P}^2$. Again j can only be a power of p, and since $\chi(p) = 0$, the only term in (1) that does not vanish is the term $j = 1$, namely $\sigma_r(N)$. Since $N = N(\mathfrak{P})^m = p^m$ and $i = 1$, this again agrees with (2).

Case 3. $\chi(p) = -1$, $(p) = \mathfrak{P}$. Now $\mathfrak{A} = \mathfrak{P}^m = (p^m)$, so j can take on the values $1, p, p^2, ..., p^m$, with $\chi(p^n) = (-1)^n$. Here $i = 2$ and $N = N(\mathfrak{P})^m = p^{2m}$, so we must prove

$$\sigma_{2r}(p^m) = \sum_{n=0}^{m} (-1)^n p^{nr} \sigma_r(p^{2m-2n}). \qquad (3)$$

This is just an exercise in summing geometric series.

The lemma enables us to calculate the generalized sums-of-powers functions $\sigma_r(\mathfrak{A})$ in terms of the ordinary function $\sigma_r(m)$. It remains to see what ideals $\mathfrak{A}$ occur in Siegel's formula. Recall that

$$s_l^k(2m) = \sum_{\substack{v \in \mathfrak{d}^{-1} \\ v \gg 0 \\ tr(v) = l}} \sigma_{2m-1}((v)\,\mathfrak{d}), \qquad (4)$$

and that

$$\mathfrak{d} = (\sqrt{D}) \qquad (5)$$

for a quadratic field. Furthermore, the ring of integers of K is

$$\mathcal{O} = \left\{ \frac{x + y\sqrt{D}}{2} \;\middle|\; x, y \in \mathbf{Z},\ x^2 \equiv y^2 D \pmod 4 \right\}. \qquad (6)$$

We can now describe explicitly the v occurring in the sum (4). Write such a v as $\alpha + \beta\sqrt{D}$ with α and β rational. Then

$$\nu \in \mathfrak{d}^{-1} \Leftrightarrow \nu\sqrt{D} \in \mathcal{O}\,, \tag{7}$$

$$\nu \gg 0 \Leftrightarrow \alpha > |\beta|\sqrt{D}\,, \tag{8}$$

$$\mathrm{tr}(\nu) = l \Leftrightarrow \alpha = l/2\,. \tag{9}$$

From (6), (7) and (9) we then get $\beta = b/2D$, where b is a rational integer satisfying

$$b^2 \equiv l^2 D \pmod 4 \tag{10}$$

and (because of (8)) also

$$b^2 < l^2 D\,. \tag{11}$$

Then $(\nu)\,\delta$ is the principal ideal

$$(\nu)\,\mathfrak{d} = (\nu\sqrt{D}) = \left(\frac{b}{2} + \frac{l}{2}\sqrt{D}\right). \tag{12}$$

An integer j can divide this only if $j \mid b$ and $j \mid l$ and $(b/j)^2 \equiv (l/j)^2 D \pmod 4$, so by the lemma

$$\sigma_r((\nu)\,\mathfrak{d}) = \sum_{\substack{l = jl' \\ b = jb' \\ b'^2 \equiv l'^2 D \ (\mathrm{mod}\ 4)}} \chi(j)\, j^r\, \sigma_r\left(\frac{l'^2 D - b'^2}{4}\right). \tag{13}$$

We now substitute this into (4), where the summation in (4) is now to be taken over all integers b satisfying (10) and (11), and obtain finally

$$s_l^K(2m) = \sum_{j \mid l} \chi(j)\, j^{2m-1}\, e_{2m-1}\left((l/j)^2 D\right), \tag{14}$$

where the arithmetic function $e_r(n)$ is defined by

$$e_r(n) = \sum_{\substack{x^2 \equiv n \ (\mathrm{mod}\ 4) \\ |x| \leqq \sqrt{n}}} \sigma_r\left(\frac{n - x^2}{4}\right) \tag{15}$$

($r = 0, 1, 2, \ldots$; n a positive integer, not a perfect square). Then (15) is a finite sum (empty, if $n \equiv 2$ or $3 \pmod 4$), and so is (14), so that we have completely evaluated $s_l^K(2m)$ in elementary terms. Then Siegel's theorem states

$$\zeta_K(1-2m) = 4 \sum_{l=1}^{r} b_l(4m)\, s_l^K(2m)\,, \tag{16}$$

with $r = [m/3]$ and the coefficients $b_l(4m)$ computable rational numbers tabulated on p. 60 for $1 \leqslant l \leqslant 10$.

Using the values of $b_l(4m)$ and equation (14), we can write out the first few cases to illustrate (16): $\underline{m = 1}$. Here $r = 1$, $b_1(4) = 1/240$, and so (16) reduces to

$$\zeta_K(-1) = \frac{1}{60}\, s_1^K(2) = \frac{1}{60}\, e_1(D)\,. \tag{17}$$

Thus for $K = \mathbf{Q}(\sqrt{5})$ we find

$$\begin{aligned}\zeta_K(-1) &= \frac{1}{60}\, e_1(5) = \frac{1}{60}\left\{\sigma_1\left(\frac{5-1^2}{4}\right) + \sigma_1\left(\frac{5-(-1)^2}{4}\right)\right\}\\ &= 2\,\sigma_1(1)/60 = 1/30\,,\end{aligned} \tag{18}$$

in agreement with 2 (17), and similarly for $K = \mathbf{Q}(\sqrt{13})$

$$\begin{aligned}\zeta_K(-1) &= \frac{1}{60}\, e_1(13) = \frac{2}{60}\left\{\sigma_1\left(\frac{13-1^2}{4}\right) + \sigma_1\left(\frac{13-3^2}{4}\right)\right\}\\ &= \frac{2}{60}(3+1+1) = 1/6\,,\end{aligned} \tag{19}$$

in agreement with 2 (18) (but notice how many fewer terms!). $\underline{m = 2}$. Here again $r = 1$, and the formula is just as simple:

$$\zeta_K(-3) = \frac{1}{120}\, s_1^K(4) = \frac{1}{120}\, e_3(D)\,. \tag{20}$$

Thus with $K = \mathbf{Q}(\sqrt{13})$ we find

$$\zeta_K(-3) = \frac{2}{120}(3^3+1^3+1^3) = \frac{29}{60}\,. \tag{21}$$

$\underline{m = 3}$. Here $r = 2$ and the formula is more complicated:

$$\begin{aligned}\zeta_K(-5) &= \frac{4}{196560}\left(s_2^K(6) - 24\, s_1^K(6)\right)\\ &= \frac{1}{49140}\{\, e_5(4D) + 32\,\chi(2)\, e_5(D) - 24\, e_5(D)\}\,.\end{aligned} \tag{22}$$

Here for $K = \mathbf{Q}(\sqrt{13})$ we get

$$\begin{aligned}\zeta_K(-5) &= (e_5(52) - 56 e_5(13))/49140\\ &= (\sigma_5(13) + 2\sigma_5(12) + 2\sigma_5(9) + 2\sigma_5(4)\\ &\quad - 112\sigma_5(3) - 112\sigma_5(1))/49140\\ &= 980370/49140 = 3631/182\,.\end{aligned} \tag{23}$$

TABLE 2.

The Siegel formulas for quadratic fields

$$K = \mathbf{Q}(\sqrt{D}), \quad D = \text{discriminant}, \quad \chi(m) = \left(\frac{D}{m}\right),$$

$$e_r(n) = \sum_{\substack{b^2+4ac=n \\ a,c>0}} a^r .$$

$60\,\zeta_K(-1) = e_1(D)$

$120\,\zeta_K(-3) = e_3(D)$

$49140\,\zeta_K(-5) = e_5(4D) + [32\,\chi(2) + 24]\,e_5(D)$

$36720\,\zeta_K(-7) = e_7(4D) + [128\,\chi(2) - 216]\,e_7(D)$

$9900\,\zeta_K(-9) = e_9(4D) + [512\,\chi(2) - 456]\,e_9(D)$

$13104000\,\zeta_K(-11) = e_{11}(9D) + 48e_{11}(4D) + [177147\,\chi(3)$
$+ 98304\,\chi(2) - 195660]\,e_{11}(D)$

$3897600\,\zeta_K(-13) = e_{13}(9D) - 192e_{13}(4D) + [1594323\,\chi(3)$
$- 1572864\,\chi(2) - 151740]\,e_{13}(D)$

$652800\,\zeta_K(-15) = e_{15}(9D) - 432e_{15}(4D) + [14348907\,\chi(3)$
$- 14155776\,\chi(2) - 50220]\,e_{15}(D)$

$1554543900\,\zeta_K(-17) = e_{17}(16D) + 72e_{17}(9D) + [131072\,\chi(2)$
$- 194184]\,e_{17}(4D) + [17179869184\,\chi(4) + 9298091736\,\chi(3)$
$- 25452085248\,\chi(2) - 57093088]\,e_{17}(D)$

$312543000\,\zeta_K(-19) = e_{19}(16D) - 168e_{19}(9D) + [524288\,\chi(2)$
$- 156024]\,e_{19}(4D) + [274877906944\,\chi(4) - 195259926456\,\chi(3)$
$- 81801510912\,\chi(2) - 19291168]\,e_{19}(D)$

$42124500\,\zeta_K(-21) = e_{21}(16D) - 408\,e_{21}(9D) + [2097152\,\chi(2)$
$- 60264]\,e_{21}(4D) + [4398046511104\,\chi(4) - 4267824106824\,\chi(3)$
$- 126382768128\,\chi(2) - 3953248]\,e_{21}(D)$

TABLE 3.

Values of $\zeta_K(1-2m)$ *for quadratic fields*

$Z_1 = 60\zeta_K(-1)$. $Z_3 = 120\zeta_K(-3)$. $Z_5 = 49140\,\zeta_K(-5)$. $Z_7 = 36720\,\zeta_K(-7)$.

$Z_9 = 9900\,\zeta_K(-9)$. $Z_{11} = 13104000\,\zeta_K(-11)$. $(K = \mathbf{Q}(\sqrt{D}),\ D = \text{discriminant})$

D	Z_1	Z_3	Z_5	Z_7	Z_9	Z_{11}
5	2	2	5226	110466	2476506	636229128800
8	5	11	70395	3765483	215478075	141611774080400
12	10	46	655590	78808158	10145592150	15002017227306400
13	10	58	1003890	143106714	21682075650	37653788862335200
17	20	164	4516980	1078232292	277803225300	823821554778449600
21	20	308	14017380	5219942004	2064025431300	9353651984246859200
24	30	522	29672370	14265873306	7346194920450	43450483506376984800
28	40	904	69359160	45338101992	31773438504600	255789968221174153600
29	30	942	82614870	58740797646	44300167762950	382856016709462960800
33	60	1692	173700540	156050858556	151482447747900	1692706573508047636800
37	50	2258	316311450	365256498834	448286221058250	6306377416787885007200
40	70	3154	493274730	658004816322	941093728561050	15461657528842738261600
41	80	3584	572460720	794742744672	1191020559229200	20543995478169063449600
44	70	4306	830983530	1344445147458	2327280476401050	46266888778260351522400

In Table 2 we write out in full the formula for $\zeta_K(1-2m)$ $(1 \leqslant m \leqslant 6)$ in terms of the arithmetical functions $e_r(n)$. In Table 3 we give the values of $\zeta_K(1-2m)$ for $1 \leqslant m \leqslant 6$ and K a quadratic field with discriminant at most 50. Since it is more convenient to tabulate integers, we in fact give the values of

$$Z_{2m-1} = t(m)\zeta_K(1-2m), \tag{24}$$

where $t(m)$ is the bound implied by (16) for the denominator of $\zeta_K(1-2m)$, namely

$$t(m) = L.C.M.\,\{\text{denom } 4b_l(4m),\ 1 \leqslant l \leqslant r\}. \tag{25}$$

Because the question of the denominator of $\zeta_K(1-2m)$ is important (namely, a prime p divides this denominator whenever the p-adic analogue of $\zeta_K(s)$ has a pole at $s = 1 - 2m$), it is worthwhile to try to sharpen (25). To do this, we use the result of §2, namely

$$\zeta_K(1-2m) = (B_{2m}/4m^2) \sum_{r=0}^{2m} B_r D^{r-1} \beta_{2m-r}(D), \tag{26}$$

where B_r is the r-th Bernoulli number and

$$\beta_r(D) = \sum_{j=1}^{D} \chi_D(j) j^r. \tag{27}$$

Set

$$a(m) = \prod_{\substack{3 \leqq p \leqq 2m+1 \\ p \text{ prime}}} p. \tag{28}$$

For $0 \leqslant r \leqslant 2m$, $2a(m)B_r$ is an integer, by von Staudt's theorem, and since $\beta_r(D) \equiv 0 \pmod 4$, $\frac{1}{2}a(m)B_r D^{r-1}\beta_{2m-r}(D)$ is an integer for $r \geqslant 1$. There remains the term $r = 0$ of (26). If D is divisible by an odd prime p but $D \neq p$, then (writing $D = pD'$, with $p \nmid D'$)

$$\beta_{2m}(D) \equiv \sum_{k=1}^{p} \chi_p(k) k^{2m} \sum_{\substack{j=1 \\ j \equiv k \ (\text{mod } p)}}^{D} \chi_{D'}(j) \pmod p, \tag{29}$$

and the inner sum is 0 for $D' > 1$. One also checks easily that $\beta_{2m}(D)$ is always even, is divisible by 8 if $D \equiv 0 \pmod 4$ and is divisible by 16 if $D \equiv 0 \pmod 8$. Therefore $\beta_{2m}(D)/D$ is an even integer, unless $D = p$ is a prime ($\equiv 1 \pmod 4$). In that case,

$$\beta_{2m}(p) = \sum_{k=1}^{p-1} \left(\frac{k}{p}\right) k^{2m} \equiv \sum_{k=1}^{p-1} k^{2m+(p-1)/2} \equiv 0 \pmod p \tag{30}$$

if $2m + \frac{p-1}{2}$ is not divisible by $p-1$. Finally, if $2m + \frac{p-1}{2}$ is divisible by $p-1$, then $(p-1) \mid 4m$ and hence $p = 4m+1$ or $p \leqslant 2m+1$. Therefore $a(m)\beta_{2m}(D)/D$ is an even integer here also, except in the one case $D = 4m+1 =$ prime. Thus, if we set

$$s(m) = a(m) \cdot \text{denom}\,(B_{2m}/2m^2) \cdot \varepsilon_m\,, \tag{31}$$

$$\varepsilon_m = \begin{cases} 4m+1 & \text{if} \quad 4m+1 \quad \text{is prime,} \\ 1 & \text{otherwise,} \end{cases} \tag{32}$$

then $s(m)\zeta_K(1-2m)$ will be an integer for all quadratic fields K, and indeed $(s(m)/\varepsilon_m)\,\zeta_K(1-2m)$ will be an integer for all fields except $\mathbf{Q}(\sqrt{4m+1})$. We have tabulated the two bounds $t(m)$ and $s(m)$ for $1 \leqslant m \leqslant 17$ in Table 4, putting the factor ε_m of $s(m)$ in brackets because it only occurs in the denominator of $\zeta_K(1-2m)$ for a single exceptional field K. It will be seen that in general neither of $s(m)$, $t(m)$ divides the other, so that

$$u(m) = G.C.D.\{\,s(m)\,,t(m)\} \tag{33}$$

gives a better bound than is provided by either the Siegel or the elementary method alone. From the table of values of $u(m)$ one sees that, for instance,

$$3 \mid Z_7\,, \quad 20 \mid Z_{11} \tag{34}$$

and that

$$5 \mid Z_1 \text{ if } D \neq 5\,, \quad 13 \mid Z_5 \text{ if } D \neq 13\,, \quad 17 \mid Z_7 \text{ if } D \neq 17\,. \tag{35}$$

All of these congruences can be verified in Table 3. Indeed, Table 3 suggests that (34) can be improved to

$$3 \mid Z_5\,, \quad 9 \mid Z_7\,, \quad 3 \mid Z_9\,, \quad 400 \mid Z_{11} \tag{36}$$

and that, as well as the congruences (35), one has

$$5 \mid Z_5\,, \quad 25 \mid Z_9 \text{ if } D \neq 5\,. \tag{37}$$

All of these are special cases of the following

CONJECTURE ([6], p. 164). *For any totally real* K,

$$w_m(K)\,\zeta_K(1-2m) \in \mathbf{Z}\,, \tag{38}$$

where the integer $w_m(K)$ *is defined as*

$$G.C.D.\,\{\,(N\mathfrak{P})^i\,(N\mathfrak{P}^{2m}-1),\ i \geqslant m,\ \mathfrak{P} \text{ a prime ideal}\}. \tag{39}$$

TABLE 4.

Bounds for the denominator of $\zeta_K(1-2m)$, *K quadratic*

m	$t(m)$ (Siegel bound)	$s(m)$ (elementary bound)	$u(m) = (t(m), s(m))$
1	$60 = 2^2 3.5$	$2^3 3^2$ (5)	$2^2 3.$ (5)
2	$120 = 2^3 3.5$	$2^5 3^2 5^2$	$2^3 3.5$
3	$49140 = 2^2 3^3 5.7.13$	$2^3 3^4 5.7^2$ (13)	$2^2 3^3 5.7.$ (13)
4	$36720 = 2^4 3^3 5.17$	$2^7 3^2 5^2 7.$ (17)	$2^4 3^2 5.$ (17)
5	$9900 = 2^2 3^2 5^2 11$	$2^3 3^2 5^2 7.11^2$	$2^2 3^2 5^2 11$
6	$13104000 = 2^7 3^2 5^3 7.13$	$2^5 3^4 5^2 7^2 11.13^2$	$2^5 3^2 5^2 7.13$
7	$3897600 = 2^8 3.5^2 7.29$	$2^3 3^2 5.7^2 11.13.$ (29)	$2^3 3.5.7.$ (29)
8	$652800 = 2^9 3.5^2 17$	$2^9 3^2 5^2 7.11.13.17^2$	$2^9 3.5^2 17$
9	$1554543900 = 2^2 3^5 5^2 7.13.19.37$	$2^3 3^6 5.7^2 11.13.17.19^2$ (37)	$2^2 3^5 5.7.13.19.$ (37)
10	$312543000 = 2^3 3^2 5^3 7.11^2 41$	$2^5 3^2 5^4 7.11^2 13.17.19.$ (41)	$2^3 3^2 5^3 7.11^2$ (41)
11	$42124500 = 2^2 3^2 5^3 11.23.37$	$2^3 3^2 5.7.11^2 13.17.19.23^2$	$2^2 3^2 5.23$
12	$141466590720 = 2^9 3^6 5.7^3 13.17$	$2^7 3^4 5^2 7^2 11.13^2 17.19.23$	$2^7 3^4 5.7^2 13.17$
13	$22877225280 = 2^6 3^5 5.7.13.53.61$	$2^3 3^2 5.7.11.13^2 17.19.23.$ (53)	$2^3 3^2 5.7.13.$ (53)
14	$2722083840 = 2^{10} 3^3 5.7.29.97$	$2^5 3^2 5^2 7^2 11.13.17.19.23.29^2$	$2^5 3^2 5.7.29$
15	$11448204768000 = 2^8 3^3 5^3 7^2 11.13.31.61$	$2^3 3^4 5^2 7^2 11^2 13.17.19.23.29.31^2$ (61)	$2^3 3^3 5^2 7^2 11.13.31.$ (61)
16	$1611414604800 = 2^{10} 3^3 5^2 7.11.13.17.137$	$2^{11} 3^2 5^2 7.11.13.17^2 19.23.29.31$	$2^{10} 3^2 5^2 7.11.13.17$
17	$176840092800 = 2^7 3^2 5^2 7.17.51599$	$2^3 3^2 5.7.11.13.17^2 19.23.29.31$	$2^3 3^2 5.7.17$

Define an integer $j(m)$ for $m = 1, 2, \ldots$ by

$$j(m) = G.C.D.\{ n^{m+2}(n^{2m}-1), n \in \mathbf{Z} \} . \tag{40}$$

Thus

$$j(1) = 24, j(2) = 240, j(3) = 504, j(4) = 480, \ldots$$

Then it is easy to check that, for K a quadratic field, $w_m(K) = j(m)$ (independent of K!) unless K is one of the finitely many fields $\mathbf{Q}(\sqrt{p})$ with p a prime such that $(p-1) \mid 4m$, $(p-1) \nmid 2m$, in which case $w_m(K)$ $p^{v+1} j(m)$, where p^v is the largest power of p dividing m. This is interesting because the numbers $j(m)$ occur in topology: it is known (now that the Adams conjecture has been proved) that $j(m)$ is precisely the order of the group $J(S^{4m})$. This may be just a coincidence, of course, but could conceivably reflect some deeper connection between the values of zeta-functions and topological K-theory (the conjectured connection between these values and algebraic K-theory was mentioned in the introduction).

§4. The Circle Method and the Numbers $e_{2m-1}(n)$

In §3 we defined

$$e_r(n) = \sum_{\substack{k^2 \equiv n \pmod 4 \\ |k| \leq \sqrt{n}}} \sigma_r\left(\frac{n-k^2}{4}\right), \tag{1}$$

where r and n are positive integers and, for b a positive integer, $\sigma_r(b)$ is defined as the sum of the r-th powers of the positive divisors of b. Since (1) was only needed for n not a perfect square, we are still at liberty to define $\sigma_r(0)$; we set

$$\sigma_r(0) = \frac{1}{2}\zeta(-r) = -\frac{1}{2}\frac{B_{r+1}}{r+1} . \tag{2}$$

This defines $\sigma_r(b)$ for $b = 0, 1, 2, \ldots$; we extend the definition to all real b by setting $\sigma_r(b) = 0$ if $b < 0$ or $b \notin \mathbf{Z}$. Then (1) can be rewritten

$$e_r(n) = \sum_{k=-\infty}^{\infty} \sigma_r\left(\frac{n-k^2}{4}\right) . \tag{3}$$

We were led to consider these numbers by Siegel's theorem, which, for real quadratic fields K, expresses the value of $\zeta_K(2m)$ or $\zeta_K(1-2m)$ in terms of the numbers $e_{2m-1}(n)$ with $K = \mathbf{Q}(\sqrt{n})$. In this section we