Afterword

Objekttyp: **Postface**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **22 (1976)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **13.09.2024**

TABLE 6

The "Hecke-Eisenstein lattice" for $m \leqslant 5$

(In the table, $Q = E_4(z)$, $R = E_6(z)$. The data for $m = 3, 4, 5$ is conjectural only.)

m	Basis for $\mathfrak{M}_{4m}^{\mathbf{Z}}$	Basis for $\mathfrak{M}_{4m}^{HE}$	$[\mathfrak{M}_{4m}^{\mathbf{Z}} : \mathfrak{M}_{4m}^{HE}]$	Exceptional discriminants
1	$\frac{1}{240} Q$	$\frac{1}{24} Q$	$2.5 = 10$	5, 8
2	$\frac{1}{480} Q^2$	$\frac{1}{240} Q^2$	2	8
3	$\frac{1}{720} Q^3$, $\frac{1}{156}\left(\frac{Q^3}{720} + \frac{R^2}{1008}\right)$	$\frac{1}{24} Q^3$, $\frac{5}{504} R^2$	$2^4.3^2.5^2.13 = 46800$	5, 8, 13
4	$\frac{1}{960} Q^4$, $\frac{1}{153}\left(\frac{Q^4}{240} + \frac{QR^2}{192}\right)$	$\frac{7}{480} Q^4$, $\frac{5}{12} QR^2$	$2^5.3^2.5.7.17 = 171360$	8, 17
5	$\frac{1}{1200} Q^5$, $\frac{1}{36}\left(\frac{Q^5}{1200} + \frac{Q^2R^2}{528}\right)$	$\frac{147}{8} Q^5$, $\frac{5}{264} Q^2R^2$	$2^4.3^4.5^3.7^2$ 7938000	5, 8

AFTERWORD

The original version of this paper was written three years ago. To bring it up to date, we must comment on two developments which have occurred in the intervening time.

1. The conjecture of Serre quoted at the end of Section 3 is now (almost) a theorem. In the original paper [6], Serre proved the partial result that,

for any totally real field K and positive integer n, $\prod_{m=1}^{n} w_m(K)\,\zeta_K(1-2m)$ is an integer (the product occurs when one calculates the "Euler characteristic" of the discrete group $Sp_{2n}\mathcal{O}$, $\mathcal{O}$ = ring of integers of K). For the case of *abelian* totally real fields (and thus in particular the case of quadratic fields), the conjecture is much easier, since it can be reduced to the evaluation of L-series, and it was proved independently by several people (e.g. J. Fresnel, "Valeurs des fonctions zêta aux entiers negatifs", *Séminaire de Théorie de Nombres*, 1970-1971, Bordeaux). In [7], Serre obtained better bounds than 3 (25), still by using Siegel's idea, but studying in more detail the p-adic behaviour of the coefficients $s_l^K(2m)$ of the Hecke-Eisenstein series. Finally, Deligne, using p-adic modular forms in several variables and a strengthened version of Mumford's results on compactifications of modular schemes (of which the details have apparently not yet been checked completely), proved Serre's conjecture for arbitrary totally real fields modulo the question of the irreducibility of a certain p-adic representation, and this question was resolved affirmatively by K. Ribet.

Related to the question of the denominator of $\zeta_K(1-2m)$ is the question of its exact fractional part (resolved for $K = \mathbf{Q}$ by the theorem of von Staudt). In connection with his work on the Hilbert modular group (*L'Enseignement Mathématique* (3-4) *19* (1973) 183-283). Hirzebruch found formulas for the fractional part of $\zeta_K(-1)$, K a real quadratic field, in terms of the class numbers of certain imaginary quadratic fields. This formula has been generalized to arbitrary totally real fields by Brown ("Euler characteristics of discrete groups and G-spaces", *Inv. Math. 27* (1974), 229-264), using the methods of [6], and by Vignéras-Guého ("Partie fractionnaire de $\zeta_K(-1)$", *C. R. Acad. Sciences, Paris* (10) *279* (1974), 359-361, "Nombres de classes d'un ordre d'Eichler et valeur au point -1 de la fonction zêta d'un corps quadratique réel", *l'Ens. Math.*, *21* (1975) 69-105) using a formula of Eichler for class numbers of orders in totally definite quaternion fields.

2. The aim of Section 4, namely to explain without the use of modular forms in two variables Siegel's formula for $\zeta_K(1-2m)$, can now be achieved in another way, both simpler and more enlightening than the application of the circle method outlined in §4. In that section, we observed that the number

$$e_{2m-1}(n) = \sum_{0 \leq n-k^2 \equiv 0 \pmod 4} \sigma_{2m-1}\left(\frac{n-k^2}{4}\right)$$

is the coefficient of $e^{\pi i n z}$ in the Fourier expansion of a function $F_m(z)$ (eq. 4 (23)) which is up to a factor the product of the ordinary theta series $\theta(z)$ and the Eisenstein series $G_{2m}(2z)$. The function $F_m(2z)$ (at least if $m > 1$) is a modular form of weight $2m + \frac{1}{2}$ for $\Gamma_0(4)$ in the sense of Shimura's paper "Modular functions of half integral weight", (*Modular Functions of One Variable* I, Lecture Notes 320, Springer Verlag, Berlin/Heidelberg/New York 1973, pp. 57-74). In this paper, Shimura discusses how to set up for such forms a theory of Hecke operators with many of the usual properties but with the essential difference that there are now Hecke operators T_n only for n a perfect square. He also shows that the two Eisenstein series of weight $2m + \frac{1}{2}$ for $\Gamma_0(4)$ have n-th Fourier coefficients related to $\zeta_{\mathbf{Q}(\sqrt{n})}(1-2m)$. In fact, one can check that there is a linear combination of these two Eisenstein series whose n-th Fourier coefficient is precisely the number

$$\bar{e}_{2m-1}(n) = \begin{cases} 0 & \text{if } n \equiv 2 \text{ or } 3 \pmod 4, \\ \dfrac{\zeta_K(1-2m)}{2\zeta(1-4m)} T^{\chi}_{2m}(f) & \text{if } n = f^2 D,\ D = \text{discriminant} \\ & \text{of } K = \mathbf{Q}(\sqrt{n}),\ \chi = (\tfrac{D}{\cdot}) \end{cases}$$

which arose in our §4 as the sum of the singular series for $e_{2m-1}(n)$. The identities of Siegel expressing $\bar{e}_{2m-1}(n)$ as a linear combination of

$$e_{2m-1}(n),\ e_{2m-1}(4n),\ e_{2m-1}(9n),\ \ldots,\ e_{2m-1}(r^2 n)\ \left(r = \left[\frac{m}{3}\right] + 1\right)$$

can now be interpreted as saying that the modular form $\sum_{n=0}^{\infty} \bar{e}_{2m-1}(n)\, e^{2\pi i n z}$ of weight $2m + \frac{1}{2}$ can be expressed as a linear combination of the function $F_m(2z)$ and its images under the Hecke operators $T_4, T_9, \ldots, T_{r^2}$. These ideas have been worked out by Cohen in three papers,

COHEN, H. Sommes de carrés, fonctions L et formes modulaires. *C. R. Acad. Sci. Paris* (A) *277* (1973), 827-830.

—— Variations sur un thème de Siegel et Hecke. *To appear in Acta Arithm. 30* (1975).

—— Sums involving the values at negative integers of L-functions of quadratic characters. *Math. Annalen 217* (1975), 271-285,

especially the last, in which he studies an arithmetic function $H(r, N)$ which is related to our function by

$$H(2m, n) = \frac{2\zeta(1-4m)}{\zeta(1-2m)} \bar{e}_{2m-1}(n).$$

However, despite these new approaches to Siegel's formula, I have retained Section 4 because the calculations of the Gauss sums $\gamma_c(n)$ and of the Dirichlet series $\sum \gamma_c(n) c^{-s}$ (Theorems 2 and 3 of §4) are often useful to have (for example, the calculation of the Fourier coefficients of the Eisenstein series of weight $2m + \frac{1}{2}$, of which is not carried out in detail in Shimura's paper, depends on them) and also because the application of the circle method in the context of forms of half-integral weight seemed novel.

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(Reçu en avril 1973, version revisée reçue le 21 juillet 1975)