§1. Introduction

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A BOUNDARY VALUE CHARACTERIZATION OF WEIGHTED H^1

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ABSTRACT

We give a proof of the elementary result that for certain weight functions w, the Hardy space H_w^1 can be identified with the class of functions f such that f and all its Riesz transforms $R_j f$ belong to L_w^1. An important ingredient of the proof is that there exist positive constants c and μ, $0 < \mu < 1$, depending only on the dimension n such that if f belongs to L_w^1, then

$$N(f)(x) \leqslant c\left[M_\mu(f)(x) + \sum_{j=1}^{n} M_\mu(R_j f)(x)\right],$$

where $N(f)$ denotes the non-tangential maximal function of the Poisson (or any conjugate Poisson) integral of f, and M_μ denotes the Hardy-Littlewood maximal operator of order μ:

$$M_\mu(g)(x) = \left(\sup_{h>0} h^{-n} \int_{|y|<h} |g(x+y)|^\mu \, dy\right)^{1/\mu}.$$

§1. INTRODUCTION

Let $F(x, t) = \left(u(x, t), v_1(x, t), \ldots, v_n(x, t)\right)$, $x = (x_1, \ldots, x_n) \in R^n$, $t > 0$, satisfy the Cauchy-Riemann equations in the sense of Stein and Weiss [9]: i.e., $u, v_1, \ldots, v_n$ are harmonic in

$$R_+^{n+1} = \{(x, t) : x \in R^n, \ t > 0\}, \quad \frac{\partial u}{\partial t} + \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j} = 0$$

and

$$\frac{\partial v_j}{\partial x_i} = \frac{\partial v_i}{\partial x_j}, \quad \frac{\partial v_j}{\partial t} = \frac{\partial u}{\partial x_j}$$

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there. F is said to belong to H^1_w, where w is a non-negative measurable function on R^n, if

$$||| F ||| = \sup_{t>0} \int_{R^n} | F(x,t) | \, w(x) \, dx < +\infty .$$

Letting

$$L^1_w = \left\{ f : \ || f ||_{1,w} = \int_{R^n} | f(x) | \, w(x) \, dx < +\infty \right\} ,$$

it is immediate that a Cauchy-Riemann system belongs to H^1_w if and only if the L^1_w norms of its components are uniformly bounded for $t > 0$. See also [4], p. 118.

We consider primarily weight functions w satisfying

$$(A_1) \qquad \frac{1}{|I|} \int_I w(x) \, dx \leqslant c \ \operatorname*{ess\,inf}_I \ w ,$$

where I is a "cube" in R^n, and c is a constant independent of I. (See [7], [3].) If $w \in A_1$ and $f \in L^1_w$, the Riesz transforms of f, defined as the *pointwise* limits

$$(1) \qquad \begin{aligned} (R_j f)(x) &= \lim_{\varepsilon \to 0} (R_{j,\varepsilon} f)(x) , \ j = 1, \dots, n , \quad \text{where} \\ (R_{j,\varepsilon} f)(x) &= c_n \int_{|y| > \varepsilon} f(x-y) \frac{y_j}{|y|^{n+1}} \, dy , \ c_n = \Gamma\left(\frac{n+1}{2}\right) / \pi^{\frac{n+1}{2}} , \end{aligned}$$

exist a.e. (See [2].) Moreover, as we shall see, the Poisson and conjugate Poisson integrals of f exist and are finite if $f \in L^1_w$, $w \in A_1$. These will be denoted respectively by

$$(Pf)(x,t) = \int_{R^n} f(x-y) \, P(y,t) \, dy ,$$

$$(Q_j f)(x,t) = \int_{R^n} f(x-y) \, Q_j(y,t) \, dy ,$$

where

$$P(y,t) = c_n t / (t^2 + |y|^2)^{\frac{n+1}{2}}$$

and

$$Q_j(y,t) = c_n y_j / (t^2 + |y|^2)^{\frac{n+1}{2}}$$

are the Poisson and conjugate Poisson kernels. The vector $(Pf, Q_1 f, \ldots, Q_n f)$ is of course a Cauchy-Riemann system, and the formulas

$$\lim_{t\to 0} (Pf)(x,t) = f(x)\,, \ \lim_{t\to 0} (Q_j f)(x,t) = (R_j f)(x)$$

hold a.e. if $f \in L^1_w$, $w \in A_1$.

THEOREM 1. *Let* $w \in A_1$.

(i) *If* $F = (u, v_1, \ldots, v_n)$ *belongs to* H^1_w, *there exists* $f \in L^1_w$ *such that* $R_j f \in L^1_w$, $u = Pf$ *and* $v_j = P(R_j f) = Q_j f$ *for each* j. *Moreover, there are positive constants* c_1 *and* c_2, *independent of* F, *such that*

$$c_1 \,|||\, F \,|||\, \leqslant \|f\|_{1,w} + \sum_1^n \| R_j f \|_{1,w} \leqslant c_2 \,|||\, F \,|||\,. \tag{2}$$

(ii) *Let* $f \in L^1_w$. *If each* $R_j f \in L^1_w$, *then the vector*

$$F = (Pf, Q_1 f, \ldots, Q_n f)$$

belongs to H^1_w. *Moreover,* $Q_j f = P(R_j f)$ *and* (2) *holds.*

Thus, if $w \in A_1$, H^1_w can be identified with

$$\{f\colon \|f\| = \|f\|_{1,w} + \sum_1^n \| R_j f \|_{1,w} < +\infty\}\,,$$

with equivalence of norms. This result, which is very natural, seems to be generally taken for granted, although there appear to be no proofs (at least of (ii)) in the literature, even when $w \equiv 1$. In the one-dimensional periodic case with $w \equiv 1$, two proofs of (ii) are given in [11], vol. 1: see (4.4), p. 263, and the remark at the bottom of p. 285. Our proof is modelled after the second of these. It is largely technical and contains little that is new; a simpler proof would be interesting. The proof of (i) is fairly standard and included only for completeness.

A weight w is said to belong to A_p, $1 < p < \infty$, if there is a constant c such that

$$\left(\frac{1}{|I|}\int_I w(x)\,dx\right)\left(\frac{1}{|I|}\int_I w(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1} \leqslant c \tag{A_p}$$

for all cubes I. (See [7]). For $0 < p < \infty$, let

$$L^p_w = \left\{f\colon \|f\|_{p,w} = \left(\int_{R^n} |f|^p w\,dx\right)^{1/p} < +\infty\right\}.$$

In the course of proving (ii), we will derive the following result, analogous to Theorem D of [9], about boundary values of Cauchy-Riemann systems.

THEOREM 2. *Let* F *be a Cauchy-Riemann system for which*

$$\sup_{t>0} \int_{R^n} |F(x,t)|^p w_1(x)\,dx < +\infty ,$$

where $\frac{n-1}{n} < p < \infty$ *and* $w_1 \in A_{pn/(n-1)}$. *Then* $F(x,t)$ *has a limit* $F(x,0)$ *a.e. (and in* $L^p_{w_1}$*) as* $t \to 0$. *If* $|F(x,0)| \in L^r_{w_2}$ *for* $\frac{n-1}{n} < r < \infty$ *and* $w_2 \in A_{rn/(n-1)}$, *there is a constant* c, *depending only on* n *and* w_1, *such that*

$$\sup_{t>0} \int_{R^n} |F(x,t)|^r w_2(x)\,dx \leqslant c\, \|F(x,0)\|^r_{r,w_2} . \tag{3}$$

The case $w_1 = w_2 = 1$ is proved in [9].

It follows (see (10) below) that for $F \in H^1_w$, $|||F|||$ and $\|F(x,0)\|_{1,w}$ are equivalent if $w \in A_{n/(n-1)}$. This is an interesting contrast to Theorem 1, which gives more boundary information, but requires the stronger condition $w \in A_1$.

The method used to prove Theorems 1 and 2 leads to the following result, in which we use the notation

$$N(F)(x) = \sup \{ |F(y,t)| : (y,t) \text{ satisfies } |x-y| < t \},$$

$$(M_\mu f)(x) = \left(\sup_{h>0} h^{-n} \int_{|y|<h} |f(x+y)|^\mu dy \right)^{1/\mu}, \mu > 0 .$$

THEOREM 3. *Let* f *belong to* L^1_w, *and let* w *satisfy* A_1. *Let* $F = (Pf, Q_1 f, \ldots, Q_n f)$. *There exist positive constants* c *and* μ *depending only on* n *such that* $0 < \mu < 1$ *and*

$$N(F)(x) \leqslant c\, [M_\mu(f)(x) + \sum_{j=1}^{n} M_\mu(R_j f)(x)] .$$

The constant μ above can be taken to be $(n-1)/n$. It follows easily from this and the results of [7] that if $f \,\varepsilon\, L^1_{w_1}$ for *any* w_1 satisfying A_1

and if $f, R_1 f, \ldots, R_n f \in L^p_{w_2}$ for some $p > \mu$ and w_2 satisfying $A_{p/\mu}$ $(= A_{pn/(n-1)})$, then $N(F) \in L^p_{w_2}$ and

$$||N(F)||_{p,w_2} \leqslant c[||f||_{p,w_2} + \sum_{j=1}^{n} ||R_j f||_{p,w_2}].$$

Finally, as a corollary of Theorem 1, we will show that if f, $R_j f \in L^1$ $(w \equiv 1)$ for all j, then the Fourier transforms satisfy the standard formula

$$(R_j f)^\wedge(x) = i \frac{x_j}{|x|} \hat{f}(x)$$

for $x \neq 0$, and, by continuity, $\widehat{(R_j f)}(0) = \hat{f}(0) = 0$. The simple proof is given at the end of §3.

§2. Preliminary results

In this section, we prove some facts, including Theorem 2, which will be useful later.

First, we need several observations about condition A_1. If g^* denotes the Hardy-Littlewood maximal function of a function g, it is not hard to see that $w \in A_1$ if and only if there is a constant c such that

(4) $$w^*(x) \leqslant c w(x) \quad \text{a.e.}$$

It is also easy to check that if $w \in A_1$ and I and J are cubes with $I \subset J$, then

(5) $$\int_J w dx \leqslant c \frac{|J|}{|I|} \int_I w dx .$$

Since for any w that is not identically zero, there is a constant $c > 0$ such that $w^*(x) \geqslant c(1+|x|)^{-n}$, we obtain that $w(x) \geqslant c(1+|x|)^{-n}$ a.e. if $w \in A_1$. Actually, if $w \in A_1$, there exists δ, $0 < \delta < 1$, such that $w^{1/\delta} \in A_1$ (see [7]), so that $w(x) \geqslant c(1+|x|)^{-n\delta}$ a.e. This shows that if $f \in L^1_w$, $w \in A_1$, then $Pf(x, t)$ and $Q_j f(x, t)$ are finite and tend to zero as $t \to +\infty$ (for fixed x). In fact, the estimate implies that

(6) $$\sup_y \frac{w(y)^{-1}}{(t+|x-y|)^n} \qquad ((x, t) \quad \text{fixed}, \quad t > 0)$$

is finite and tends to zero as $t \to +\infty$. Thus, since $P(x-y, t)$ and $Q_j(x-y, t)$ are bounded in absolute value by a multiple of $(t+|x-y|)^{-n}$, it follows that $|Pf(x, t)|$ and $|(Q_j f)(x, t)|$ are bounded by