and if $f, R_1 f, \ldots, R_n f \in L^p_{w_2}$ for some $p > \mu$ and w_2 satisfying $A_{p/\mu}$ $(= A_{pn/(n-1)})$, then $N(F) \in L^p_{w_2}$ and

$$||N(F)||_{p,w_2} \leqslant c[||f||_{p,w_2} + \sum_{j=1}^{n} ||R_j f||_{p,w_2}].$$

Finally, as a corollary of Theorem 1, we will show that if f, $R_j f \in L^1$ $(w \equiv 1)$ for all j, then the Fourier transforms satisfy the standard formula

$$(R_j f)^\wedge(x) = i \frac{x_j}{|x|} \hat{f}(x)$$

for $x \neq 0$, and, by continuity, $(R_j f)^\wedge(0) = \hat{f}(0) = 0$. The simple proof is given at the end of §3.

§2. Preliminary results

In this section, we prove some facts, including Theorem 2, which will be useful later.

First, we need several observations about condition A_1. If g^* denotes the Hardy-Littlewood maximal function of a function g, it is not hard to see that $w \in A_1$ if and only if there is a constant c such that

$$w^*(x) \leqslant c\, w(x) \quad \text{a.e.} \tag{4}$$

It is also easy to check that if $w \in A_1$ and I and J are cubes with $I \subset J$, then

$$\int_J w\,dx \leqslant c \frac{|J|}{|I|} \int_I w\,dx\,. \tag{5}$$

Since for any w that is not identically zero, there is a constant $c > 0$ such that $w^*(x) \geqslant c(1+|x|)^{-n}$, we obtain that $w(x) \geqslant c(1+|x|)^{-n}$ a.e. if $w \in A_1$. Actually, if $w \in A_1$, there exists δ, $0 < \delta < 1$, such that $w^{1/\delta} \in A_1$ (see [7]), so that $w(x) \geqslant c(1+|x|)^{-n\delta}$ a.e. This shows that if $f \in L^1_w$, $w \in A_1$, then $Pf(x, t)$ and $Q_j f(x, t)$ are finite and tend to zero as $t \to +\infty$ (for fixed x). In fact, the estimate implies that

$$\sup_y \frac{w(y)^{-1}}{(t+|x-y|)^n} \qquad ((x, t) \quad \text{fixed}, \quad t > 0) \tag{6}$$

is finite and tends to zero as $t \to +\infty$. Thus, since $P(x-y, t)$ and $Q_j(x-y, t)$ are bounded in absolute value by a multiple of $(t+|x-y|)^{-n}$, it follows that $|Pf(x, t)|$ and $|(Q_j f)(x, t)|$ are bounded by

$$c \int_{R^n} |f(y)| \frac{dy}{(t+|x-y|)^n} \leqslant c \, \|f\|_{1,w} \left\{ \sup_y \frac{w(y)^{-1}}{(t+|x-y|)^n} \right\},$$

which is finite and tends to zero as $t \to +\infty$.

In addition to the pointwise existence of $R_j f$ for $f \in L^1_w$, $w \in A_1$, there is also a weak-type estimate: if $m_w(E)$ denotes the w-measure of a set E (i.e., $m_w(E) = \int_E w\, dx$) and if $R_j^* f$ is defined by

$$(R_j^* f)(x) = \sup_{\varepsilon > 0} |(R_{j,\varepsilon} f)(x)|,$$

then

$$m_w\{x\colon (R_j^* f)(x) > \lambda\} \leqslant c\lambda^{-1} \|f\|_{1,w}, \ \lambda > 0,$$

with c independent of f and λ. A similar estimate holds for f^*. (See [2], [7].)

We need several facts about condition $A_p, p > 1$, all of which can be found in [2], [6], and [7]. Here we note only that if $w \in A_p, p > 1$, there is a constant c such that

$$(B_p) \qquad \int_{R^n} \frac{w(y)}{(t+|x-y|)^{np}} dy \leqslant ct^{-np} \int_{|x-y|<t} w(y)\, dy, \ t > 0.$$

(Cf. lemma 1 of [6].) In particular, $w(y)/(1+|y|)^{np}$ is integrable over R^n if $w \in A_p$. This shows that Pf and $Q_j f$ are finite if $f \in L^p_w$, $w \in A_p, p > 1$. In fact, by Hölder's inequality,

$$\int_{R^n} |f(y)| \frac{dy}{(t+|x-y|)^n} \leqslant \|f\|_{p,w} \left(\int_{R^n} \frac{w(y)^{-p'/p}}{(t+|x-y|)^{np'}} dy \right)^{1/p'},$$

$p' = p/(p-1)$. Since $w \in A_p$, we have $w^{-p'/p} \in A_{p'}$, so that

$$w(y)^{-p'/p}/(1+|y|)^{np'}$$

is integrable and the last expression is finite.

We need the following lemma about harmonic majorization.

LEMMA 1. *Let $s(x,t)$ be subharmonic in R^{n+1}_+ and satisfy*

$$\sup_{t>0} \int_{R^n} |s(x,t)|^p w(x)\, dx < +\infty$$

for some p, $1 \leqslant p < +\infty$, with $w \in A_p$. Then for $a > 0$,

$$(7) \qquad s(x, t+a) \leqslant P(s(\cdot, a))(x,t).$$

If s is harmonic, equality holds in (7).

Proof. First note by the remarks above that $P(s(.,a))(x,t)$ is finite, since $s(.,a) \in L^p_w$, $w \in A_p$. Inequality (7) is a corollary of Theorem 2 of [8], provided that we show

$$(a) \qquad \sup_{t>0} \int_{R^n} \frac{|s(x,t)|}{(1+t+|x|)^{n+1}}\, dx < +\infty\,,$$

$$(b) \qquad \lim_{t\to+\infty} \int_{R^n} \frac{|s(x,t)|}{(1+t+|x|)^{n+1}}\, dx = 0\,.$$

If $p > 1$,

$$\int_{R^n} \frac{|s(x,t)|}{(1+t+|x|)^{n+1}}\, dx$$

$$\leqslant \left(\int_{R^n} |s(x,t)|^p w(x)\, dx\right)^{1/p} \left(\int_{R^n} \frac{w(x)^{-p'/p}}{(1+t+|x|)^{(n+1)p'}}\, dx\right)^{1/p'}$$

$$\leqslant c \left(\int_{R^n} \frac{w(x)^{-p'/p}}{(1+t+|x|)^{(n+1)p'}}\, dx\right)^{1/p'}.$$

Since $(1+t+|x|)^{(n+1)p'} \geqslant (1+t)^{p'}(1+|x|)^{np'}$ and $w^{-p'/p}$ satisfies $B_{p'}$, the last expression is at most

$$\frac{c}{1+t}\left(\int_{R^n} \frac{w(x)^{-p'/p}}{(1+|x|)^{np'}}\, dx\right)^{1/p'} \leqslant \frac{c}{1+t}\left(\int_{|x|<1} w(x)^{-p'/p}\, dx\right)^{1/p'},$$

from which (a) and (b) follow. The argument for $p = 1$ is similar, using for example the simple estimate $w(x)^{-1} \leqslant c(1+|x|)^n$. Finally, if s is harmonic then $s(x, t+a) = P(s(.,a))(x,t)$, by applying (7) to both s and $-s$.

LEMMA 2. *Let F be a Cauchy-Riemann system for which*

$$\sup_{t>0} \int_{R^n} |F(x,t)|^p w(x)\, dx < +\infty\,,$$

where $\dfrac{n-1}{n} < p < \infty$ *and* $w \in A_{pn/(n-1)}$. *Then* $F(x,t)$ *converges a.e. to a limit* $F(x,0)$ *as* $t \to 0$. *Moreover,* $\|F(x,t) - F(x,0)\|_{p,w} \to 0$ *as* $t \to 0$, *and there is a constant* c *depending only on* n *such that*

$$(8) \qquad N(F)(x) \leqslant c\left(|F(x,0)|^{\frac{n-1}{n}}\right)^{*\,\frac{n}{n-1}},$$

where $*$ *denotes the Hardy-Littlewood maximal function.*

Proof. Except for the last estimate, this lemma is proved in [4]. The method is standard. Let $q = pn/(n-1)$ and $s(y, t) = |F(y, t)|^{\frac{n-1}{n}} = |F(y, t)|^{\frac{p}{q}}$. Then s is non-negative, continuous and (by [9]) subharmonic in R_+^{n+1}. Also,

$$\int_{R^n} s(y, t)^q w(y)\, dy = \int_{R^n} |F(y, t)|^p w(y)\, dy \leqslant c_1, \quad t > 0.$$

Since $q > 1$, there exist $\{t_k\} \to 0$ and $h \in L_w^q$ such that $\|h\|_{q,w}^q \leqslant c_1$ and $s(., t_k)$ converges weakly in L_w^q to h—i.e.,

$$\int_{R^n} s(y, t_k)\, g(y)\, w(y)\, dy \to \int_{R^n} h(y)\, g(y)\, w(y)\, dy$$

if $g \in L_w^{q'}$, $q' = q/(q-1)$. For fixed (x, t), choose $g(y) = P(x-y, t)\, w(y)^{-1}$. Since $w \in A_q$, we have $w^{-q'/q}$ $(= w^{1-q'}) \in B_{q'}$, and therefore $g \in L_w^{q'}$. For this g, the integral on the left above equals $P(s(., t_k))(x, t)$, which majorizes $s(x, t+t_k)$ by Lemma 1, and the integral on the right equals $(Ph)(x, t)$. Hence,

$$s(x, t) = \lim_{t_k \to 0} s(x, t+t_k) \leqslant (Ph)(x, t).$$

Therefore, $|F(x, t)| \leqslant (Ph)(x, t)^{\frac{q}{p}}$, so that

$$N(F)(x) \leqslant N(h)(x)^{\frac{q}{p}} \leqslant ch^*(x)^{\frac{q}{p}}. \tag{9}$$

We have

$$\int_{R^n} h^{*q} w\, dx \leqslant c \int_{R^n} |h|^q w\, dx$$

by [7]. Hence, h^*, and so $N(F)$, is finite a.e., and it follows from [1] that F has non-tangential boundary values $F(x, 0)$ a.e. Moreover,

$$\int_{R^n} |F(x, t) - F(x, 0)|^p w(x)\, dx \to 0 \quad \text{as} \quad t \to 0$$

by dominated convergence:

$$|F(x, t)| \leqslant N(F)(x) \leqslant ch^*(x)^{q/p} \in L_w^p.$$

Since $s(x,t) = |F(x,t)|^{\frac{n-1}{n}}$, we have $s(x,t) \to |F(x,0)|^{\frac{n-1}{n}}$ a.e. This convergence is also in L_w^q norm since

$$| s(x,t) | \leqslant N(F)(x)^{\frac{p}{q}} \in L_w^q .$$

Since $s(.,t_k)$ also converges weakly in L_w^q to h, it follows that $h(x) = |F(x,0)|^{\frac{n-1}{n}}$ a.e. Inequality (8) now follows immediately from (9).

Proof of Theorem 2. Let F be a Cauchy-Riemann system satisfying

$$\sup_{t>0} \int_{R^n} | F(x,t) |^p w_1(x)\, dx < +\infty ,$$

where

$$\frac{n-1}{n} < p < \infty , \quad w_1 \in A_{pn/(n-1)} .$$

Then F has boundary value $F(x,0)$ a.e. and in L_w^p by Lemma 2; moreover,

$$N(F)(x) \leqslant c\,(|F(x,0)|^{\frac{n-1}{n}})^{*\frac{n}{n-1}} .$$

If we now assume that

$$| F(x,0) | \in L_{w_2}^r , \quad \frac{n-1}{n} < r < \infty , \quad w_2 \in A_{rn/(n-1)} ,$$

then

$$\int_{R^n} N(F)(x)^r w_2(x)\, dx \leqslant c \int_{R^n} (|F(x,0)|^{\frac{n-1}{n}})^{*\frac{nr}{n-1}}\, w_2(x)\, dx$$

$$\leqslant c \int_{R^n} | F(x,0) |^r w_2(x)\, dx$$

by [7]. This gives (3) immediately.

Remark. We note in passing that if

$$\sup_{t>0} \int_{R^n} | F(x,t) |^p w(x)\, dx < +\infty , \quad \frac{n-1}{n} < p < \infty , \quad w \in A_{pn/(n-1)} ,$$

then

$$\text{(10)} \qquad \sup_{t>0} \int_{R^n} | F(x,t) |^p w(x)\, dx \approx \| F(\cdot,0) \|_{p,w}^p .$$

This follows from Theorem 2: the right-hand side is at most a multiple of the left since $F(x, t) \to F(x, 0)$ in L^p_w; the converse inequality is just (3) with w_2 and r chosen to be w and p, resp.

§3. Proof of Theorems 1 and 3

We will prove Theorem 1 first, beginning with part (i). Let $F \in H^1_w$, $F = (u, v_1, \ldots, v_n)$, $w \in A_1$. By Theorem 2, F has boundary values $F(x, 0) = \big(f(x), g_1(x), \ldots, g_n(x)\big)$ pointwise a.e. and in L^1_w. In particular, $f, g_1, \ldots, g_n \in L^1_w$. We will show that $u = P(f)$ and $v_j = P(g_j)$. Since $u(x, s)$ converges to $f(x)$ in L^1_w, $P\big(u(.,s)\big)(x, t) \to (Pf)(x, t)$ as $s \to 0$:

$$|P\big(u(\cdot, s)\big)(x, t) - (Pf)(x, t)| = |\int_{R^n} [u(y, s) - f(y)] P(x-y, t)\, dy|$$

$$\leqslant \|u(\cdot, s) - f\|_{1,w} \{\sup_y w(y)^{-1} P(x-y, t)\},$$

where the expression in curly brackets is finite for each (x, t) (see (6)). By Lemma 1, $u(x, s+t) = P\big(u(.,s)\big)(x, t)$ since u is harmonic. Hence, letting $s \to 0$, we obtain $u(x, t) = (Pf)(x, t)$, as desired. The argument proving that $v_j = P(g_j)$ is similar.

Now let $G = (Pf, Q_1 f, \ldots, Q_n f)$. Then G is a Cauchy-Riemann system with the same first component as F. This implies that the first component of F-G is zero, and so that the others are independent of t; that is, $v_j - Q_j f$ is independent of t. Thus, $v_j = Q_j f$ if both $v_j(x, t)$ and $(Q_j f)(x, t)$ tend to zero as $t \to +\infty$ (x fixed). We have already observed this for $Q_j f$. For v_j, the mean-value property of harmonic functions gives

$$|v_j(x, t)| \leqslant ct^{-n-1} \iint_{|\xi - x|^2 + |t-\eta|^2 < t^2} |v_j(\xi, \eta)|\, d\xi d\eta$$

$$\leqslant ct^{-n} \sup_{\eta > 0} \int_{|\xi - x| < t} |v_j(\xi, \eta)|\, d\xi$$

$$\leqslant ct^{-n} \left(\sup_{\eta > 0} \int_{R^n} |v_j(\xi, \eta)|\, w(\xi)\, d\xi\right) \Big(\sup_{\xi:\ |\xi - x| < t} w(\xi)^{-1}\Big)$$

$$\leqslant ct^{-n} \sup_{\xi : |\xi - x| < t} w(\xi)^{-1}.$$