Zeitschrift: L'Enseignement Mathématique
Band: 22 (1976)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: BOUNDARY VALUE CHARACTERIZATION OF WEIGHTED H^1
Kapitel: §3. Proof of Theorems 1 and 3
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DOI: https://doi.org/10.5169/seals-48178

This follows from Theorem 2: the right-hand side is at most a multiple of the left since $F(x, t) \to F(x, 0)$ in L^p_w; the converse inequality is just (3) with w_2 and r chosen to be w and p, resp.

§3. Proof of Theorems 1 and 3

We will prove Theorem 1 first, beginning with part (i). Let $F \in H^1_w$, $F = (u, v_1, \ldots, v_n)$, $w \in A_1$. By Theorem 2, F has boundary values $F(x, 0) = \big(f(x), g_1(x), \ldots, g_n(x)\big)$ pointwise a.e. and in L^1_w. In particular, $f, g_1, \ldots, g_n \in L^1_w$. We will show that $u = P(f)$ and $v_j = P(g_j)$. Since $u(x, s)$ converges to $f(x)$ in L^1_w, $P\big(u(., s)\big)(x, t) \to (Pf)(x, t)$ as $s \to 0$:

$$| P\big(u(\cdot, s)\big)(x, t) - (Pf)(x, t) | = | \int_{R^n} [u(y, s) - f(y)] P(x-y, t)\, dy |$$

$$\leqslant || u(\cdot, s) - f ||_{1,w} \{ \sup_y w(y)^{-1} P(x-y, t) \},$$

where the expression in curly brackets is finite for each (x, t) (see (6)). By Lemma 1, $u(x, s+t) = P\big(u(., s)\big)(x, t)$ since u is harmonic. Hence, letting $s \to 0$, we obtain $u(x, t) = (Pf)(x, t)$, as desired. The argument proving that $v_j = P(g_j)$ is similar.

Now let $G = (Pf, Q_1 f, \ldots, Q_n f)$. Then G is a Cauchy-Riemann system with the same first component as F. This implies that the first component of F-G is zero, and so that the others are independent of t; that is, $v_j - Q_j f$ is independent of t. Thus, $v_j = Q_j f$ if both $v_j(x, t)$ and $(Q_j f)(x, t)$ tend to zero as $t \to +\infty$ (x fixed). We have already observed this for $Q_j f$. For v_j, the mean-value property of harmonic functions gives

$$| v_j(x, t) | \leqslant c t^{-n-1} \iint_{|\xi - x|^2 + |t-\eta|^2 < t^2} | v_j(\xi, \eta) |\, d\xi d\eta$$

$$\leqslant c t^{-n} \sup_{\eta > 0} \int_{|\xi - x| < t} | v_j(\xi, \eta) |\, d\xi$$

$$\leqslant c t^{-n} \left(\sup_{\eta > 0} \int_{R^n} | v_j(\xi, \eta) |\, w(\xi)\, d\xi \right) \left(\sup_{\xi :\ |\xi - x| < t} w(\xi)^{-1}\right)$$

$$\leqslant c t^{-n} \sup_{\xi : |\xi - x| < t} w(\xi)^{-1}.$$

Since $w(\xi)^{-1} \leqslant c(1+|\xi|)^{n\delta}$ for some δ, $0 < \delta < 1$, we have

$$|v_j(s,t)| \leqslant ct^{-n}(1+|x|+t)^{n\delta}.$$

Hence, $v_j(x,t) \to 0$ as $t \to \infty$ for each x.

We now know $u = Pf$, $v_j = P(g_j) = Q_j f$. Letting $t \to 0$ in the equation $P(g_j)(x,t) = (Q_j f)(x,t)$ gives $g_j(x) = (R_j f)(x)$ a.e. Thus, $R_j f \in L^1_w$ and $v_j = P(R_j f) = Q_j f$, as desired. All that remains to prove in (i) is that $|||\, F \,|||$ and $\|f\|_{1,w} + \sum_{j=1}^{n} \|R_j f\|_{1,w}$ are equivalent. This, however, follows immediately from (10) with $p = 1$, since

$$F(x,0) = \big(f(x), R_1 f(x), \ldots, R_n f(x)\big).$$

To prove (ii), let f be a function in L^1_w for which each $R_j f \in L^1_w$. (The existence of $R_j f$ as a pointwise limit is guaranteed by the hypothesis $w \in A_1$.) We will show that the vector defined by

$$F = (Pf, Q_1 f, \ldots, Q_n f)$$

is in H^1_w. Once this is done, the rest of (ii) clearly follows from (i). We know F is a Cauchy-Riemann system, and only need to show $|||\, F \,||| < +\infty$. As $t \to 0$, $F(x,t)$ converges a.e. to $(f, R_1 f, \ldots, R_n f) = F(x,0)$, say, so that $|F(x,0)| \in L^1_w$. Hence, $|||\, F \,||| < +\infty$ by Theorem 2 if there exist p and w_1, $\dfrac{n-1}{n} < p < \infty$, $w_1 \in A_{pn/(n-1)}$, such that

(11) $$\sup_{t>0} \int_{R^n} |F(x,t)|^p w_1(x)\,dx < +\infty.$$

We first claim that if $w \in A_1$, there exists $\alpha > 0$ such that the function

$$w_1(x) = \frac{w(x)}{(1+|x|)^\alpha}$$

also belongs to A_1. Note that $(1+|x|)^{-\beta} \in A_1$ if $0 \leqslant \beta < n$, and that there exists $s > 1$ such that $w^s \in A_1$. Hence, for any cube I, Hölder's inequality gives

$$\frac{1}{|I|}\int_I w_1(x)\,dx \leqslant \left(\frac{1}{|I|}\int_I w(x)^s\,dx\right)^{1/s}\left(\frac{1}{|I|}\int_I (1+|x|)^{-\alpha s'}\,dx\right)^{1/s'},$$

$s' = s/(s-1)$. Choose $\alpha > 0$ so small that $\alpha s' < n$. Then both w^s and $(1+|x|)^{-\alpha s'}$ are in A_1, and

$$\frac{1}{|I|}\int_I w_1(x)\,dx \leqslant c\,(\operatorname*{ess\,inf}_I w^s)^{1/s}(\operatorname*{ess\,inf}_I(1+|x|)^{-\alpha s'})^{1/s'}$$
$$= c\,(\operatorname*{ess\,inf}_I w)\,(\operatorname*{ess\,inf}_I(1+|x|)^{-\alpha})$$
$$\leqslant c \operatorname*{ess\,inf}_I w_1 .$$

This proves the claim.

With this choice of w_1, we will complete the proof of (ii) by showing that (11) holds for any $p < 1$ which is sufficiently close to 1. Let

$$(R^*f)(x) = \max_{j=1,\dots,n} (R_j^*f)(x) .$$

Then, as is well-known, there is a constant c depending only on n such that

$$|F(x,t)| \leqslant c\,[f^*(x) + (R^*f)(x)] .$$

It follows from the weak-type estimates referred to in §2 that the radial maximal function $N_0(F)(x)\ \big(= \sup_{t>0} |F(x,t)|\big)$ satisfies

$$m_w\{x\colon N_0(F)(x) > \lambda\} \leqslant c\lambda^{-1}\,||f||_{1,w},\ \lambda > 0 .$$

We will show that any non-negative function ϕ with

$$m_w\{x\colon \phi(x) > \lambda\} \leqslant c\lambda^{-1},\ \lambda > 0 ,$$

belongs to $L^p_{w_1}$, $1 - \dfrac{\alpha}{n} < p < 1$. Let $g_r(\lambda)$, $\lambda > 0$, denote the non-increasing rearrangement of a function g with respect to the measure $w(x)\,dx$. Then, by [5], p. 257,

$$\int_{R^n} \phi^p w_1 dx = \int_{R^n} \phi(x)^p (1+|x|)^{-\alpha}\, w(x)\,dx$$
$$\leqslant \int_0^\infty \phi_r^p(\lambda)\,\{(1+|x|)^{-\alpha}\}_r(\lambda)\,d\lambda .$$

We have $\phi_r(\lambda) \leqslant c\,\lambda^{-1}$ and must estimate $\{(1+|x|)^{-\alpha}\}_r$. However,

$$m_w\{x\colon (1+|x|)^{-\alpha} > \lambda\} = m_w\{x\colon 1+|x| < \lambda^{-1/\alpha}\} ,$$

which for $\lambda \geqslant 1$ is zero and for $0 < \lambda < 1$ is less than

$$\int_{|x|<\lambda^{-1/\alpha}} w\,dx \leqslant c\lambda^{-n/\alpha} \int_{|x|<1} w\,dx = c\lambda^{-n/\alpha}$$

(see (5)). Therefore,

$$\{(1+|x|)^{-\alpha}\}_r(\lambda) \leqslant c(1+\lambda)^{-\alpha/n}, \ \lambda > 0 .$$

Combining estimates, we obtain

$$\int_{R^n} \phi^p w_1 dx \leqslant c \int_0^\infty \lambda^{-p}(1+\lambda)^{-\alpha/n} d\lambda < +\infty$$

if $1 - \frac{\alpha}{n} < p < 1$, as desired. This completes the proof of (ii).

To prove Theorem 3, let $f \in L^1_w$ and $w \in A_1$. Then (11) holds for F, p and w_1 as in the proof of Theorem 1 (ii). (The proof of (11) does not require $R_j f \in L^1_w$.) Hence, by Lemma 2 (see (8)),

$$N(F)(x) \leqslant c(|F(x,0)|^{\frac{n-1}{n}})^{*\frac{n}{n-1}} .$$

Since $F(x,0) = \big(f(x), (R_1 f)(x), \ldots, (R_n f)(x)\big)$, the conclusion of Theorem 3 follows immediately with $\mu = (n-1)/n$.

To prove the fact stated at the end of the introduction, let

$$f, R_1 f, \ldots, R_n f \in L^1 .$$

Clearly,

$$P(R_j f)\hat{\ }(x,t) = \hat{P}(x,t)(R_j f)\hat{\ }(x) = e^{-2\pi t|x|}(R_j f)\hat{\ }(x),$$

$$(Q_j f)\hat{\ }(x,t) = \hat{Q}_j(x,t)\hat{f}(x) = i\frac{x_j}{|x|}e^{-2\pi t|x|}\hat{f}(x) \text{ a.e.},$$

where the Fourier transform is taken in the x variable with t fixed. (Note that for fixed t, $P(x,t)$ belongs to L^1 and $Q_j(x,t)$ belongs to L^2.) However, these expressions are all equal everywhere since $P(R_j f) = Q_j f$ by Theorem 1 and $P(R_j f) \in L^1$. Therefore, $(R_j f)\hat{\ }(x) = ix_j|x|^{-1}\hat{f}(x)$, as claimed.

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