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(3)  $E$  has split multiplicative reduction at 3  $\Leftrightarrow a_1^2 + a_2 \equiv 1 \pmod{3}$ .

(4)  $E$  has non-split multiplicative reduction at 3  $\Leftrightarrow a_1^2 + a_2 \equiv -1 \pmod{3}$ .

*Proof:*

$$c_4 \equiv b_2^2 - 24b_4 \equiv b_2^2 \equiv (a_1^2 + 4a_2)^2 \equiv (a_1^2 + a_2)^2 \pmod{3}.$$

The theorem then follows immediately from formula (3.1) and Corollary 1.2.

*Remark.*  $C_2^2 \equiv c_4 \pmod{3}$ . Note that  $C_2 = a_1^2 + a_2$  is a more sensitive invariant than  $c_4$  in that the residue class of  $C_2$  modulo 3 allows us to distinguish between split and non-split multiplicative reduction, while  $c_4$  does not allow us to separate these two possibilities.

#### §4. THE CASE $p \geq 5$

Assume  $p \geq 5$ . Then there exists a minimal Weierstrass equation for  $E$  at  $p$  of the form

$$(4.1) \quad Y^2 = X^3 + AX + B$$

with  $A, B \in \mathbf{Z}$ . The coefficient  $C_{p-1}$  modulo  $p$  is given by Deuring's classical formula [1]

$$(4.2) \quad C_{p-1} \equiv \sum_{2h+3i=P} \frac{P!}{i! h! (P-h-i)!} A^h B^i \pmod{p}$$

where  $P = (1/2)(p-1)$ .

Let  $S = (x, y)$  be the singular point on the reduced curve with  $x, y \in \mathbf{Z}/p\mathbf{Z}$ . The tangents at  $S$  are given by a quadratic polynomial  $R(T)$  as follows: Transform the curve by  $X \rightarrow (X+x)$ ,  $Y \rightarrow (Y+y)$  so that the singularity is now at  $(0, 0)$ . The tangents are given by a homogeneous form of degree 2 in  $X$  and  $Y$  which we can consider as a quadratic polynomial

$R(T)$  with  $T = Y/X$ . Let  $D$  be the discriminant of  $R(T)$ , and let  $\left(\frac{-}{p}\right)$

denote the Legendre symbol with respect to  $p$ . We have the following results directly from the definitions.

**PROPOSITION 4.1.** Assume  $E$  has bad reduction at  $p$ .

(1)  $E$  has additive reduction at  $p \Leftrightarrow f_p = 0 \Leftrightarrow S$  is a cusp  $\Leftrightarrow R(T)$  has two identical roots over  $\mathbf{Z}/p\mathbf{Z} \Leftrightarrow D = 0 \Leftrightarrow \left(\frac{D}{p}\right) = 0$ .

(2)  $E$  has split multiplicative reduction at  $p \Leftrightarrow f_p = 1 \Leftrightarrow S$  is a node with rational tangents  $\Leftrightarrow R(T)$  has two distinct roots rational over  $\mathbf{Z}/p\mathbf{Z} \Leftrightarrow \left(\frac{D}{p}\right) = 1$ .

(3)  $E$  has non-split multiplicative reduction at  $p \Leftrightarrow f_p = -1 \Leftrightarrow S$  is a node with irrational tangents  $\Leftrightarrow R(T)$  has two distinct roots not rational over  $\mathbf{Z}/p\mathbf{Z} \Leftrightarrow \left(\frac{D}{p}\right) = -1$ .

COROLLARY 4.2.  $f_p = \left(\frac{D}{p}\right)$ .

In this case,  $H$  reduces to

$$(4.3) \quad H = Y^2 - X^3 - AX - B$$

Then we have

$$(4.4) \quad \partial H / \partial X = -3X^2 - A$$

$$(4.5) \quad \partial H / \partial Y = 2Y$$

From (4.5) we must have  $y = 0$ . From (4.4) we must have  $x^2 = -A/3$  in  $\mathbf{Z}/p\mathbf{Z}$ , so that  $-A/3$  is either a quadratic residue modulo  $p$  or 0 modulo  $p$ . Note that  $x = 0 \Leftrightarrow A \equiv 0 \pmod{p}$ . Let  $X^3 + AX + B = (X - \alpha_1)(X - \alpha_2)(X - \alpha_3)$  be a factorization over  $\mathbf{Z}/p\mathbf{Z}$ . At least two of  $\alpha_1, \alpha_2, \alpha_3$  must coincide with  $x$ , let us say  $x = \alpha_2 = \alpha_3$ . Then

$$(4.6) \quad X^3 + AX + B = X^3 + (-\alpha_1 - 2\alpha_2)X^2 + (2\alpha_1\alpha_2 + \alpha_2^2)X - \alpha_1\alpha_2^2$$

Thus comparing coefficients, we have

$$(4.7) \quad 0 = -\alpha_1 - 2\alpha_2$$

$$(4.8) \quad A = 2\alpha_1\alpha_2 + \alpha_2^2$$

$$(4.9) \quad B = -\alpha_1\alpha_2^2$$

Hence

$$(4.10) \quad \alpha_1 = -2\alpha_2$$

$$(4.11) \quad A = 2\alpha_1\alpha_2 + \alpha_2^2 = -3\alpha_2^2 = -3x^2$$

$$(4.12) \quad B = -\alpha_1\alpha_2^2 = 2\alpha_2^3 = 2x^3$$

From (4.12) we see that  $B/2$  is either a cubic residue modulo  $p$  or 0 modulo  $p$ . Note that  $x = 0 \Leftrightarrow B \equiv 0 \pmod{p}$  from (4.12).

Transform the curve by  $X \rightarrow (X + \alpha_2)$ ,  $Y \rightarrow Y$  so that the singular point  $S = (x, y) = (x, 0) = (\alpha_2, 0)$  goes to  $(0, 0)$ . We obtain

$$(4.13) \quad Y^2 - (X + \alpha_2)^3 - A(X + \alpha_2) - B = Y^2 - X^3 - 3\alpha_2 X^2$$

The tangents to  $(0, 0)$  on the transformed curve are given by

$$(4.14) \quad Y^2 - 3\alpha_2 X^2 = 0$$

so that the polynomial  $R(T)$  is  $R(T) = T^2 - 3\alpha_2$ .  $D = 12\alpha_2 = 12x$ .

$$c_4 = b_2^2 - 24b_4 = (a_1^2 + 4a_2)^2 - 24(a_1 a_3 + 2a_4) = -48A.$$

Since

$$x = 0 \Leftrightarrow A \equiv 0 \pmod{p}, \quad D = 0 \Leftrightarrow A \equiv 0$$

and so the invariant  $c_4$  is enough to distinguish between additive and multiplicative reduction. However, as we shall see below it does not separate split and non-split multiplicative reduction.

**THEOREM 4.3.** Assume that  $E$  has bad reduction at  $p$ .

(1)  $E$  has additive reduction at  $p \Leftrightarrow A \equiv 0 \pmod{p} \Leftrightarrow B \equiv 0 \pmod{p}$

$$\Leftrightarrow \left( \frac{-2AB}{p} \right) = 0.$$

(2)  $E$  has split multiplicative reduction at  $p \Leftrightarrow \left( \frac{-2AB}{p} \right) = 1$ .

(3)  $E$  has non-split multiplicative reduction at  $p \Leftrightarrow \left( \frac{-2AB}{p} \right) = -1$ .

*Proof:* (1) We have seen that  $A \equiv 0 \pmod{p} \Leftrightarrow x = 0 \Leftrightarrow B \equiv 0 \pmod{p}$ .  $E$  has additive reduction at  $p \Leftrightarrow D = 12x = 0 \Leftrightarrow x = 0$

$$\Leftrightarrow A \equiv B \equiv 0 \pmod{p} \Leftrightarrow \left( \frac{-2AB}{p} \right) = 0.$$

(2) and (3). Assume  $E$  has multiplicative reduction at  $p$ . Then  $3\alpha_2 \neq 0$ . From (4.14) we see that  $E$  has split multiplicative reduction at  $p \Leftrightarrow 3\alpha_2$  is a square in  $\mathbf{Z}/p\mathbf{Z}$ . From formulas (4.11) and (4.12) we have that  $3\alpha_2 = (-9/2)B/A$ . Thus  $3\alpha_2$  is a square  $\Leftrightarrow (-9/2)B/A$  is a square modulo  $p$

$$\Leftrightarrow -2AB \text{ is a square modulo } p \Leftrightarrow \left( \frac{-2AB}{p} \right) = 1.$$

**COROLLARY 4.4.**  $f_p = \left( \frac{-2AB}{p} \right)$ .