FINITE GEOMETRIES IN THE THEORY OF THETA CHARACTERISTICS

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INTRODUCTION

The aim of this paper is to call attention upon the existence of a very simple "finite geometry" on the set of either odd or even theta characteristics (on an algebraic curve), and to develop on some of its properties and related concepts. In particular, this finite geometry allows one to place in a general context the classical theory of the 28 bitangents to a plane quartic (cf. Weber [6]).

Part I of the paper recalls the several interpretations and definitions of theta characteristics, and contains some examples to motivate the abstract developments in Part II. In this later part, the finite geometry is defined and its properties discussed. The main result is theorem II 2.6. Proposition II 4.4 is also of important practical value.

It is my feeling that the finite geometries will be of help in studying such problems as: relations between theta functions, filtrations in the space of moduli of level two structures over curves of a given genus, degeneration of algebraic curves. A sequel to this paper should contain applications to these subjects.

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I. THETA CHARACTERISTICS ON AN ALGEBRAIC CURVE

§ 0 REVIEW: QUADRATIC FORMS IN CHARACTERISTIC 2

In this section, a number of well-known results on quadratic forms in characteristic two are recalled.

0.1 *Alternate forms.* Let J be a finite-dimensional vector space over $\mathbf{Z}/2\mathbf{Z}$, $e: J \times J \to \mathbf{Z}/2\mathbf{Z}$ a non-degenerate, alternate, bilinear form. Recall that non-degenerate means that e makes J its own dual, i.e. that the induced mapping $J \to J^*$ is a bijection; alternate signifies that the equation

$$e(x, x) = 0$$

is valid throughout J. It can then be proved that there exists a basis $x_1, \dots, x_g, x_1', \dots, x_g'$ for J such that

$$e(x_i, x_j) = e(x_i', x_j') = 0 .$$
$$e(x_i, x_j') = \delta_{ij} ,$$

in particular that the dimension of J is even. Such a basis is called a *symplectic basis* for (J, e). The *symplectic group* for (J, e), written $\mathrm{Sp}(J, e)$, is the group of linear automorphisms of J compatible with e, i.e. linear automorphisms $\sigma: J \to J$ such that for any $x, y \in J$

$$e(x, y) = e\big(\sigma(x), \sigma(y)\big) .$$

The symplectic group acts on the set of symplectic basis for (J, e), and clearly in a simply transitive way. A datum of the form (J, e) will be called a *symplectic pair* for short.

0.2 *Quadratic forms.* Let J be a finite-dimensional vector space over $\mathbf{Z}/2\mathbf{Z}$. A *quadratic form* on J is a mapping $q: J \to \mathbf{Z}/2\mathbf{Z}$ with the property that the mapping

$$e_q(x, y) = q(x) + q(y) + q(x+y)$$

is bilinear. It is clear that e_q is also alternate.

Let e be a fixed non-degenerate, alternate, bilinear form on J. There always is some quadratic form q on J such that $e_q = e$, for example

$$q(x) = \Sigma\, \lambda_i \lambda_i'$$

where $x = \Sigma\, \lambda_i x_i + \Sigma\, \lambda_i x_i'$ in terms of some symplectic basis $x_1, \dots, x_g, x_1', \dots, x_g'$ for (J, e). Moreover, if $Q(J, e)$ is the set of quadratic forms q with the property $e_q = e$, the group J acts on it through the formula

$$(x+q)(y) = q(y) + e(x, y) , \quad x, y \in J ,$$

and clearly in a simply transitive way; note that the action is written additively.

0.3 *Arf invariant.* Let (J, e) be a symplectic pair, and let $x_1, \ldots, x_g$, $x'_1, \ldots, x'_g$ be a symplectic basis for (J, e). If $q \in Q(J, e)$, it is easily proved that the scalar $\Sigma\, q(x_i)\, q(x'_i)$ is independent of the given symplectic basis; this is called the *Arf invariant* of q and will be written $Q_e(q)$. It is a mapping

$$Q_e: Q(J, e) \to \mathbf{Z}/2\mathbf{Z}$$

and it has the following property, that can be easily checked:

$$Q_e(q) + Q_e(x+q) + Q_e(y+q) + Q_e(x+y+q) = e(x, y)$$

where $x, y \in J$, $q \in Q(J, e)$, and the action defined at the end of 0.2 is being used.

The Arf invariant has the following meaning: if $q \in Q(J, e)$, the set $q^{-1}(0)$ has either $2^{g-1}(2^g+1)$ or $2^{g-1}(2^g-1)$ elements, and correspondingly $q^{-1}(1)$ has either $2^{g-1}(2^g-1)$ or $2^{g-1}(2^g+1)$ elements, where $2g = \dim J$; the first (resp. the second) happens iff $Q_e(q)$ equals 0 (resp. 1).

It is not difficult to prove that the set $Q_e^{-1}(0)$ of elements of $Q(J, e)$ with Arf invariant zero has order $2^{g-1}(2^g+1)$ and correspondingly that $Q_e^{-1}(1)$ has $2^{g-1}(2^g-1)$ elements.

0.4 *Functoriality.* Let (J, e), (J', e') be two symplectic pairs, and let $\sigma: J \to J'$ be a linear isomorphism compatible with e, e', i.e. verifying

$$e'(\sigma(x), \sigma(y)) = e(x, y) \quad x, y \in J .$$

The isomorphism σ induces a mapping

$$Q(\sigma): Q(J, e) \to Q(J', e')$$

defined by the formula

$$Q(\sigma)(q) = q \cdot \sigma^{-1} ,$$

and this has the property

$$Q(\sigma)(x+q) = \sigma(x) + Q(\sigma)(q) \quad x \in J, q \in Q(J, e) .$$

Moreover, $Q(\sigma)$ is compatible with the Arf invariant mappings $Q_{e'}$, $Q_{e'}$, in the sense that one has

$$Q_{e'} \cdot Q(\sigma) = Q_e .$$

0.5 *The standard situation.* For a given natural number g (the "genus"), let $J_o = (\mathbf{Z}/2\mathbf{Z})^{2g}$, e_o be defined by the matrix

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

where $0, I$ are respectively the zero, identity $g \times g$ matrix. The datum (J_o, e_o) is a symplectic pair, and is standard in the sense that for a fixed (J, e), giving a symplectic basis for (J, e) amounts to the same thing as giving a linear isomorphism $J_o \simeq J$ compatible with e_o, e. By 0.4, this in turn defines an isomorphism $Q(J_o, e_o) \simeq Q(J, e)$ with the properties stated there.

Going back to the standard situation, there is an obvious identification $Q(J_o, e_o) \simeq (\mathbf{Z}/2\mathbf{Z})^{2g}$, obtained associating with every quadratic form q its values on the canonical basis of J_o. With this identification in mind, the action of J_o on $Q(J_o, e_o)$ defined at the end of 0.2 is the action of $(\mathbf{Z}/2\mathbf{Z})^{2g}$ on itself by translations, and the Arf invariant is given by the mapping Q: $(\varepsilon, \varepsilon') \mapsto \Sigma\, \varepsilon_i\, \varepsilon_i'$, where $\varepsilon, \varepsilon' \in (\mathbf{Z}/2\mathbf{Z})^g$.

We will use the following notation,

$$\begin{aligned} J_0(g) &= (\mathbf{Z}/2\mathbf{Z})^{2g} \\ S_0(g) &= Q(J_0, e_0) \\ S_0^+(g) &= \{s \in S_0(g) / Q(s) = 0\} \\ S_0^-(g) &= \{s \in S_0(g) / Q(s) = 1\} \end{aligned}$$

§ 1 Theta characteristics

1.1 *On an algebraic curve.* Let C be a non-singular projective algebraic curve over an algebraically closed base field k of characteristic different from 2. The set $S(C)$ of *theta characteristics* on C is the set of isomorphism classes of line bundles L on C whose tensor square is isomorphic to the canonical bundle. If $J_2(C)$ is the group of points of order two in Pic (C), i.e. the multiplicative group of isomorphism classes of line bundles on C whose square is the trivial line bundle $\mathbf{O}_C$, then clearly $J_2(C)$ acts on the set $S(C)$, and this in a simply transitive way. In addition, there is a function

$$Q : S(C) \rightarrow \mathbf{Z}/2\mathbf{Z}$$

defined by

$$Q(L) = \dim \Gamma(C, L) \quad (2).$$

The following formula holds, where $x, y \in J_2(C)$, $s \in S(C)$, and we use additive notation both for the group law in $J_2(C)$ and the action of $J_2(C)$ on $S(C)$:

$$Q(s) + Q(x+s) + Q(y+s) + Q(x+y+s) = e(x, y).$$

Here, e stands for the intersection pairing on $J_2(C)$. If g is the genus of C, it is proved that $Q^{-1}(0)$ (resp. $Q^{-1}(1)$) has $2^{g-1}(2^g+1)$ (resp. $2^{g-1}(2^g-1)$) elements.

The proof of these assertions goes back to Riemann in the case $k = \mathbf{C}$, and in the general case it may be found in Mumford [5].

1.2 *On a principally polarized abelian variety*. Let X be an abelian variety over k, $\theta\colon X \xrightarrow{\sim} \hat{X}$ a principal polarization. The set $S(X, \theta)$ of *theta characteristics* on (X, θ) is the subset of $\mathrm{Pic}^\theta(X)$ determined by the symmetric line bundles; i.e. the elements of $S(X, \theta)$ are the isomorphism classes of line bundles L on X belonging to θ and such that $L \simeq i^*(L)$, where $i\colon X \to X$ sends $x \in X$ into $-x$. Again, the group X_2 of points of order two in X acts on $S(X, \theta)$ through the induced isomorphism $\theta\colon X_2 \xrightarrow{\sim} \hat{X}_2$, and this in a simply transitive way. Now, for any symmetric line bundle L on X, there exists a unique isomorphism $\varphi\colon L \xrightarrow{\sim} i^*(L)$ such that over the zero of X, φ induces the identity on the fibers. Over any $x \in X_2$, the fibers of L, $i^*(L)$ identify naturally, and φ induces the multiplication by some scalar that will be denoted $e_*^L(x)$. It is proved that $e_*^L(x) = \pm 1$, and indeed that $e_*^L\colon X_2 \to \mathbf{Z}/2\mathbf{Z}$ is a quadratic form whose associated bilinear form is the intersection pairing e on X_2. Now we define a mapping

$$Q\colon S(X, \theta) \to \mathbf{Z}/2\mathbf{Z}$$

by

$$Q(s) = \text{Arf invariant of } e_*^s\,.$$

The following formula is valid, where additive notation is used both for group law and group action, and where $s \in S(X, \theta)$, $x, y \in X_2$

$$Q(s) + Q(x+s) + Q(y+x) + Q(x+y+s) = e(x, y)\,.$$

It is also true that, if $g = \dim X$, $Q^{-1}(0)$ (resp. $Q^{-1}(1)$) has $2^{g-1}(2^g+1)$ (resp. $2^{g-1}(2^g-1)$) elements.

All the preceding is proved in or follows easily from § 2 of Mumford [4] and from § 0 above. Note in addition that in $\mathrm{Pic}^{2\theta}(X)$ there is a unique totally symmetric line bundle L_o (i.e. L_o is symmetric and $e_*^{L_o}(x) = 1$ for every $x \in X_2$), and that the symmetric line bundles in $\mathrm{Pic}^\theta(X)$ are the line bundles L such that L^2 is isomorphic with L_o (cf. Mumford [4], *loc. cit.*).

§ 2 Relation with the classical notation

Throughout this section the base field is **C**.

2.1 *Jacobians.* I recall briefly the data associated with a nonsingular projective curve C. We have two abelian varieties, the Jacobian variety $J(C) = H^{1,0}(C)^*/H_1(C, Z)$ and the Picard variety $P^o(C) = H^{0,1}(C)/\mathrm{H}^1(C, Z)$. From standard dualities it turns out that $P^o(C)$ is naturally isomorphic to the dual Jacobi variety $J(C)^\wedge$, and from Abel's theorem it results that there is in addition a natural isomorphism $P^o(C) \simeq J(C)$. Thus, we have associated with C a principally polarized abelian variety that I will denote henceforth $P^o(C)$, θ_C and will be called the Picard or the Jacobi variety of C according to taste. If we visualize $P^o(C)$ as the group of line bundles on C with Chern class zero, we are led to introduce the family of sets $P^h(C)$, where $P^h(C)$ is the set of isomorphism classes of line bundles with Chern class equal to $h \in \mathbf{Z}$. Each of the sets $P^h(C)$ is a torsor under $P^o(C)$, i.e. is acted on by $P^o(C)$ in a simply transitive way.

There is a natural embedding

$$C \to P^1(C)$$

and it can be proved that this induces an isomorphism of $P^o(C)$-torsors

$$\mathrm{Pic}^\theta P^1(C) \xrightarrow{\sim} P^g(C) \tag{2.1.1}$$

where $\mathrm{Pic}^\theta P^1(C)$ is the set of line bundles P on $P^1(C)$ belonging to θ, and g is the genus of C (see next section 2.2). Observe that $\mathrm{Pic}^\theta P^1(C)$ is properly a $P^o(C)^\wedge$-torsor, but it becomes a $P^o(C)$-torsor through the polarization θ.

2.2 *A simple formalism.* Let X be an abelian variety, P and X-torsor such that the group action $X \times P \to P$ be analytic. Then there are canonical isomorphisms

$$H^i(X, \mathbf{Z}) \simeq H^i(P, \mathbf{Z})$$
$$H^i(X, \mathbf{O}_x) \simeq H^i(P, \mathbf{O}_p)$$

and in particular

$$NS(X) \simeq NS(P)\,\mathrm{Pic}^o(X) \simeq \mathrm{Pic}^o(P)\,.$$

This is because the translations induce the identity both in $H^i(X, \mathbf{Z})$, $H^i(X, \mathbf{O}_x)$ as it may be easily seen. Recall that the Néron-Severi group of X (resp. of P) is the quotient

$$NS(X) = \text{Pic}(X)/\text{Pic}^o(X),$$

or also the kernel of the homomorphism

$$H^2(X, \mathbf{Z}) \to H^2(X, \mathbf{O}_x).$$

Now let $\theta\colon X \to \hat{X}$ be a polarization, θ corresponds naturally to an element $\theta \in NS(X)$, and the set $\text{Pic}^\theta(X)$ of isomorphism classes of line bundles on X belonging to θ is the coset of $\hat{X}$ in $\text{Pic}(X)$ corresponding to θ (cf. for example, Mumford, Abelian Varieties). Thus, $\text{Pic}^\theta(P)$ is well defined too, since $NS(P)$ and $NS(X)$ identify.

Starting from (X, θ) and P we have the following situation. The set $\text{Pic}^\theta(P)$ is a torsor over $\text{Pic}^o(P)$, but $\text{Pic}^o(P)$ identifies naturally with $\hat{X}$, thus $\text{Pic}^\theta(P)$ is an $\hat{X}$-torsor. The following formula makes explicit this $\hat{X}$-torsor as tensor product (the natural operation between torsors over a fixed abelian group) of two other $\hat{X}$-torsors, $\text{Pic}^\theta X$ and the $\hat{X}$-torsor $P \otimes_X \hat{X}$ obtained from P through the extension of scalars $\theta\colon X \to \hat{X}$.

(2.2.1) $$\text{Pic}^\theta(P) \simeq \text{Pic}^\theta(X) \otimes (P \otimes_X \hat{X})$$

To have this natural isomorphism it is enough to define an X-equivariant pairing $\text{Pic}^\theta(X) \times P \to \text{Pic}^\theta(P)$ and this is the obvious one: if $L \in \text{Pic}^\theta(X)$, $p \in P$ and if $t_p\colon X \to P$ is the isomorphism $t_p(x) = p + x$, then the pairing associates with (L, p) the line bundle $(t_p)_*(L)$.

This isomorphism will be used in the next section.

2.3 *Relation between 1.1, 1.2.* Let C be a nonsingular projective algebraic curve, $(P^o(C), \theta_C)$ its Picard variety with its principal polarization. Then, the definitions of theta characteristics of 1.1, 1.2 applied respectively to C, $(P^o(C), \theta_C)$ yield objects that identify naturally. Indeed, if follows from (2.1.1) and (2.2.1) that for any $h \in Z$ there is a natural isomorphism of $P^o(C)$-torsors.

$$\text{Pic}^\theta(P^h(C)) \simeq P^{h+g-1}(C),$$

where g is the genus of C. In particular, we have isomorphisms

$$\text{Pic}^\theta(P^o(C)) \simeq P^{g-1}(C)$$
$$\text{Pic}^{2\theta}(P^o(C)) \simeq P^{2g-2}(C).$$

In the last one it is easily seen that the canonical bundle corresponds to the unique totally symmetric bundle in $\text{Pic}^{2\theta} P^o(C)$. As the symmetric bundles in $\mathrm{Pic}^{\theta}(P^{o}(C))$ are exactly the square roots of this totally symmetric line bundle, it follows that $S(C)$, $S(P^{o}(C), \theta_C)$ identify naturally. Moreover, this identification is compatible with their structures of $J_2(C)$-torsors and with the maps $Q: S(C) \to \mathbf{Z}/2\mathbf{Z}$, $Q: S(P^{o}(C), \theta_C) \to \mathbf{Z}/2\mathbf{Z}$. This last point follows easily from proposition 2 in § 2 of Mumford [4] and from the theorem of Riemann (see Fay [2], theorem 1.1) stating that for a line bundle $L \in P^{g-1}(C)$, the dimension of $\Gamma(C, L)$ equals the multiplicity of the theta divisor at the point L. (In fact, observe that the theta divisor as an element of $\mathrm{Pic}^{\theta}(P^{g-1}(C))$ corresponds to the canonical bundle on C under the isomorphism $\mathrm{Pic}^{\theta}(P^{g-1}(C)) \approx P^{2g-2}(C)$.

2.4 *Theta functions*. Let (X, θ) be a principally polarized abelian variety. There is a canonical isomorphism

$$X \simeq H^{1,0}(X)^*/H_1(X, \mathbf{Z})$$

and the principal polarization corresponds to a nondegenerate alternate bilinear pairing

$$\theta: H_1(X, \mathbf{Z}) \, x \, H_1(X, \mathbf{Z}) \to \mathbf{Z}\,.$$

Let $x_1, \ldots, x_g, x_1', \ldots, x_g'$ be a symplectic basis for θ on $H_1(X, Z)$; then the images of $x_1', \ldots, x_g'$ in $H^{1,0}(X)^*$ constitute a basis for this $\mathbf{C}$-vector space, and let $w_1, \ldots, w_g$ be its dual basis for $H^{1,0}(X)$. In other words,

$$\int_{x_i'} w_j = \delta_{ij}\,.$$

Then the matrix $\tau = (\tau_{ij})$ defined by

$$\tau_{ij} = \int_{x_i} w_j$$

belongs to the Siegel upper-half space of degree g, i.e. τ is symmetric and $Im(\tau)$ is positive definite. The choice of the symplectic basis sets an identification

$$X \simeq \mathbf{C}^g/(\tau\mathbf{Z}^g \oplus \mathbf{Z}^g)\,.$$

We may now consider the classical theta functions (Igusa [3])

$$\theta_{mm^*}(\tau, z) = \sum_{\xi \in \mathbf{Z}^g} \mathbf{e}\left[\tfrac{1}{2}(\zeta + m)\,\tau^t(\zeta + m) + (\zeta + m)^t(z + m^*)\right].$$

By the properties of these theta functions and through the preceding identification, each $\theta_{mm^*}(\tau, -)$ defines a line bundle on X, and indeed an element of $\mathrm{Pic}^{\theta}(X)$ that is independent of $(m, m^*) \in \mathbf{R}^{2g} \bmod \mathbf{Z}^{2g}$. In this way we get a bijection

$$\mathrm{Pic}^{\theta}(X) \simeq \mathbf{R}^{2g}/\mathbf{Z}^{2g}\,.$$

It follows from formula (θ. 1) in p. 49 of Igusa [3] that the subset of $\mathrm{Pic}^{\theta}(X)$ defined by the symmetric line bundles on X corresponds to the image in $\mathbf{R}^{2g}/\mathbf{Z}^{2g}$ of $\frac{1}{2}\mathbf{Z}^{2g}$.

We finally see that the symplectic basis on $H_1(X, Z)$ defines an identification

$$S(X, \theta) \simeq (\mathbf{Z}/2\mathbf{Z})^{2g}.$$

It is easy to see that this identification depends only on the symplectic basis induced on

$$H_1(X, \mathbf{Z})/2H_1(X, \mathbf{Z}) \simeq H_1(X, \mathbf{Z}/2\mathbf{Z}),$$

and that it is compatible with the identification

$$\hat{X}_2 \simeq H_1(X, \mathbf{Z}/2\mathbf{Z}) \simeq (\mathbf{Z}/2\mathbf{Z})^{2g}$$

that the later basis defines and with the respective action of $\hat{X}_2$ on $S(X, \theta)$ and of $(\mathbf{Z}/2\mathbf{Z})^{2g}$ on itself by translations.

2.5 *Summing up.* If C is a nonsingular projective algebraic curve of genus g, there are two equivalent ways of defining the set of theta characteristics, either directly as in 1.1, or through its Picard variety as in 1.2. The set of theta characteristic is endowed with a simply transitive action of the group $J_2(C)$ and with a function $Q\colon S(C) \to \mathbf{Z}/2\mathbf{Z}$ closely related to the intersection pairing e on $J_2(C)$. Also, we know that $Q^{-1}(0)$ has $2^{g-1}(2^g+1)$ elements and $Q^{-1}(1)$ has $2^{g-1}(2^g-1)$ elements. Indeed, there is a third way of defining the set of theta characteristics, namely as the set $Q\big(J_2(C), e\big)$ of all quadratic forms g on $J_2(C)$ whose associated bilinear form is e; we saw in § 0 that on this set there is a structure of the same type as in $S(C)$, $S(X, \theta)$, and in fact $S(X, \theta)$ is clearly isomorphic with $Q(\hat{X}_2, e) \simeq Q(X_2, e)$.

Now if we choose a symplectic basis $x_1, \ldots, x_g, x_1', \ldots, x_g'$ for $J_2(C)$, the set $S(C)$ identifies with $(\mathbf{Z}/2\mathbf{Z})^{2g}$. In particular, $0 \in (\mathbf{Z}/2\mathbf{Z})^{2g}$ defines a "base" theta characteristic. In terms of quadratic forms, this identification corresponds to the one discussed in 0.5, in particular the base theta characteristic is even (i.e. belongs to $Q^{-1}(0)$) and it corresponds to the quadratic form q_o defined by $q_o(x_i) = q_o(x_i') = 0$. for $i = 1, \ldots, g$. Looking at $S(C)$ as a subset of $P^{g-1}(C)$, the base theta characteristic is nothing else that the Riemann constant Δ in the non-intrinsic version of the Riemann theorem referred to at the end of 2.3. (See theorem 1.1 in Fay [2] and its corollary 1.5).

§ 3 SOME SPECIAL CASES

I present here some examples in order to motivate the general discussion in Part II. Proofs of most assertions are omitted and they may be found in or follow easily from Part II. The base field is **C** to simplify things.

3.1 *Genus two.* Let C be of genus two, and let P_C be the projective space of hyperplanes in $H^{1,0}(C)$. Then P_C is a projective line, and the natural map $C \to P_C$ presents C as a 2-sheeted covering of P_C ramified over a subset $R_C \subset P_C$ with $|R_C| = 6$. From the Riemann-Roch theorem it may be proved that the line bundles L in $S(C)$ with $Q(L) = 1$, i.e. the odd theta characteristics, are those represented by effective divisors, and from here it follows easily that the set $S(C)$ of odd theta characteristics identifies naturally with R_C. If s_1, s_2, s_3 are three different elements of $S^-(C)$ represented by line bundles L_1, L_2, L_3, it is also easily proved that $L_1 \oplus L_2 \oplus L_3^{-1}$ is even. From this, and from II 2.4 it follows that there is a natural group isomorphism

$$\mathrm{Sp}(H_1(C, \mathbf{Z}/2\mathbf{Z})) \simeq \mathrm{Aut}(R_C).$$

It follows also from *loc. cit.* that it amounts to the same thing to give a symplectic basis for $H_1(C, \mathbf{Z}/2\mathbf{Z})$ or to give a bijection $S_0^-(2) \simeq R_C$, where $S_0^-(2)$ is the fixed 6-elements set defined in 0.5.

I will discuss $S^+(C)$ in a more general setting:

3.2 *Even genus, hyperelliptic case.* Let C be hyperelliptic. Then there is a projective line P_C and a map $C \to P_C$ defined up to unique isomorphisms such that $C \to P_C$ is a 2-sheeted covering. If R_C is the ramification locus, $|R_C| = 2g + 2$, and R_C identifies naturally with the set of Weierstrass points of C.

The group $H_1(C, \mathbf{Z}/2\mathbf{Z})$ can be reconstructed starting from R_C in the following way. If $\pi = \{\pi', \pi''\}$ is any partition of R_C into two even-order subsets, L_π is the line bundle defined by the divisor $\sum_{P' \in \pi_1'} P - \sum_{P \in \pi_2''} P$ where $|\pi_1'| = |\pi_2'|$ and $\{\pi_1', \pi_2'\}$ partition π'. It is clear that L_π is of order two, thus defining an element of $H_1(C, \mathbf{Z}/2\mathbf{Z})$. In this manner one gets a group isomorphism

$$P_2^+(R_C) \xrightarrow{\sim} H_1(C, \mathbf{Z}/2\mathbf{Z})$$

where the group $P_2^+(R_C)$ is defined in II 3.5. It is easily verified that this isomorphism is compatible with the intersection pairing on H_1 and with the alternated bilinear form introduced in *loc. cit.*

All the preceding was valid for any genus g. Now if g is even, it follows from II 3.6 and II 1.4 that we have an isomorphism

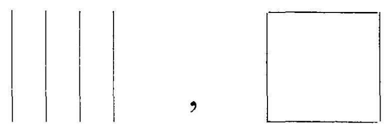
$$P_2^-(R_C) \xrightarrow{\sim} S(C)$$

compatible with the structures involved (i.e. an isomorphism of symplectic torsors, cf. II 1.1). The results of II, § 3 may thus be applied to the study of $S(C)$.

Observe that if g is odd, there is a natural theta characteristic; namely, the line bundle of the divisor $(g-1)P$ is independent of the Weierstrass point P (compare II 3.6b)).

3.3 *Genus three.* Two cases arise for C of genus three:

3.3.1 *C hyperelliptic.* Then there is the 2-sheeted covering $C \to P_C$ ramified over R_C with $|R_C| = 8$. It is seen in this case, as in 3.1, that there is a natural identification between $S^-(C)$ and the set of subsets of R_C consisting of exactly two elements. It is convenient to visualize the elements of $S^-(C)$ as segments joining the points of R_C, these being distributed on a plane in an arbitrary way. Then, if s_1, s_2, s_3, s_4 are four different elements of $S^-(C)$, $s_1 - s_2 = s_3 - s_4$ iff the segments corresponding to them produce one of the following configurations



From II 2.7 it follows that there is a canonical isomorphism between the group $Sp(H_1(C, \mathbf{Z}/2\mathbf{Z}))$ and the group of permutations of the set $S^-(C)$ that preserve the "geometry" defined by these quadruples. Two comments are in order:

a) Although the permutation group Aut (R_C) is clearly a subgroup of the automorphism group of the "geometry", not every such automorphism arises from a permutation of R_C.

b) The automorphisms of the geometry do not preserve the type of the configuration, they may send one quadruple of the first type drawn above into the other. However in a continuous family of *hyperelliptic* curves of genus 3, each of the two configurations will be preserved as the curve is deformed.

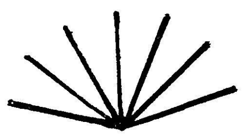
3.3.2 *C non hyperelliptic.* Let $Q_C = \mathbf{P}(H^{1,0}(C))$ be the projective space of hyperplanes in $H^{1,0}(C)$. Then Q_C is a projective plane and the natural map $C \to Q_C$ is an immersion. The degree of C in Q_C is the degree of the canonical bundle, i.e. 4 and C is thus a *nonsingular plane quartic.* It is again a simple exercise to prove that the odd theta characteristics on C correspond to the set of lines in Q_C that are bitangents to C. Thus, if B_C is the set of bitangents to C in Q_C, there is a natural identification

$$B_C \simeq S^-(C).$$

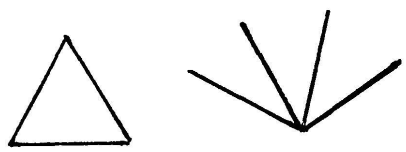
The theme of the 28 bitangents to a nonsingular plane quartic ($28 = 2^{3-1}(2^3-1)$) is a classic one in geometry, see for instance Weber [6], chapter 12. A triple (s_1, s_2, s_3) of bitangents is called *syzygetic* (resp. *azygetic*) if their six points of contact with C lie (resp. do not lie) in a conic. A triple is syzygetic iff $L_4 = L_1 \otimes L_2 \otimes L_3^{-1}$ is an odd characteristic, where L_1, L_2, L_3 are the line bundles corresponding to s_1, s_2, s_3. When this happens, the two points of contact of the bitangent s_4 corresponding to L_4, together with the preceding six, make up the full $8 = 2 \times 4$ common points of the conic with the quartic.

An *Aronhold system* of bitangents (Weber [6]) is a set of seven bitangents such that any different three of them constitute an azygetic triple. The Aronhold systems are exactly the basis for the "geometry" in $S^-(C)$ defined by the syzygetic triples (in the sense of II 4.3). It follows from II 4.4 that the set of Aronhold systems is a torsor over the symplectic group $Sp(H_1(C, \mathbf{Z}/2\mathbf{Z}))$, in particular that they have the same number of elements.

As any two "geometries" with the same genus are isomorphic (II 1.4), one can also speak of Aronhold systems in the hyperelliptic case. It turns out that they correspond to the following configurations



and



There are 1,451,520 of them as it is "immediately" checked. Again, it will be observed that the automorphisms of the geometry do not preserve the type of the configuration.