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FINITE GEOMETRIES IN THE THEORY OF THETA CHARACTERISTICS

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INTRODUCTION

The aim of this paper is to call attention upon the existence of a very simple "finite geometry" on the set of either odd or even theta characteristics (on an algebraic curve), and to develop on some of its properties and related concepts. In particular, this finite geometry allows one to place in a general context the classical theory of the 28 bitangents to a plane quartic (cf. Weber [6]).

Part I of the paper recalls the several interpretations and definitions of theta characteristics, and contains some examples to motivate the abstract developments in Part II. In this later part, the finite geometry is defined and its properties discussed. The main result is theorem II 2.6. Proposition II 4.4 is also of important practical value.

It is my feeling that the finite geometries will be of help in studying such problems as: relations between theta functions, filtrations in the space of moduli of level two structures over curves of a given genus, degeneration of algebraic curves. A sequel to this paper should contain applications to these subjects.

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I. THETA CHARACTERISTICS ON AN ALGEBRAIC CURVE

§ 0 REVIEW: QUADRATIC FORMS IN CHARACTERISTIC 2

In this section, a number of well-known results on quadratic forms in characteristic two are recalled.

0.1 *Alternate forms.* Let J be a finite-dimensional vector space over $\mathbf{Z}/2\mathbf{Z}$, $e: J \times J \to \mathbf{Z}/2\mathbf{Z}$ a non-degenerate, alternate, bilinear form. Recall that non-degenerate means that e makes J its own dual, i.e. that the induced mapping $J \to J^*$ is a bijection; alternate signifies that the equation

$$e(x, x) = 0$$

is valid throughout J. It can then be proved that there exists a basis $x_1, \ldots, x_g, x_1', \ldots, x_g'$ for J such that

$$e(x_i, x_j) = e(x_i', x_j') = 0 .$$
$$e(x_i, x_j') = \delta_{ij} ,$$

in particular that the dimension of J is even. Such a basis is called a *symplectic basis* for (J, e). The *symplectic group* for (J, e), written $\mathrm{Sp}(J, e)$, is the group of linear automorphisms of J compatible with e, i.e. linear automorphisms $\sigma: J \to J$ such that for any $x, y \in J$

$$e(x, y) = e(\sigma(x), \sigma(y)) .$$

The symplectic group acts on the set of symplectic basis for (J, e), and clearly in a simply transitive way. A datum of the form (J, e) will be called a *symplectic pair* for short.

0.2 *Quadratic forms.* Let J be a finite-dimensional vector space over $\mathbf{Z}/2\mathbf{Z}$. A *quadratic form* on J is a mapping $q: J \to \mathbf{Z}/2\mathbf{Z}$ with the property that the mapping

$$e_q(x, y) = q(x) + q(y) + q(x+y)$$

is bilinear. It is clear that e_q is also alternate.

Let e be a fixed non-degenerate, alternate, bilinear form on J. There always is some quadratic form q on J such that $e_q = e$, for example

$$q(x) = \Sigma \lambda_i \lambda_i'$$

where $x = \Sigma \lambda_i x_i + \Sigma \lambda_i x_i'$ in terms of some symplectic basis $x_1, \ldots, x_g$, $x_1', \ldots, x_g'$ for (J, e). Moreover, if $Q(J, e)$ is the set of quadratic forms q with the property $e_q = e$, the group J acts on it through the formula

$$(x+q)(y) = q(y) + e(x, y) , \quad x, y \in J ,$$

and clearly in a simply transitive way; note that the action is written additively.

0.3 *Arf invariant*. Let (J, e) be a symplectic pair, and let $x_1, \ldots, x_g$, $x_1', \ldots, x_g'$ be a symplectic basis for (J, e). If $q \in Q(J, e)$, it is easily proved that the scalar $\Sigma\, q(x_i)\, q(x_i')$ is independent of the given symplectic basis; this is called the *Arf invariant* of q and will be written $Q_e(q)$. It is a mapping

$$Q_e : Q(J, e) \to \mathbf{Z}/2\mathbf{Z}$$

and it has the following property, that can be easily checked:

$$Q_e(q) + Q_e(x+q) + Q_e(y+q) + Q_e(x+y+q) = e(x, y)$$

where $x, y \in J$, $q \in Q(J, e)$, and the action defined at the end of 0.2 is being used.

The Arf invariant has the following meaning: if $q \in Q(J, e)$, the set $q^{-1}(0)$ has either $2^{g-1}(2^g+1)$ or $2^{g-1}(2^g-1)$ elements, and correspondingly $q^{-1}(1)$ has either $2^{g-1}(2^g-1)$ or $2^{g-1}(2^g+1)$ elements, where $2g = \dim J$; the first (resp. the second) happens iff $Q_e(q)$ equals 0 (resp. 1).

It is not difficult to prove that the set $Q_e^{-1}(0)$ of elements of $Q(J, e)$ with Arf invariant zero has order $2^{g-1}(2^g+1)$ and correspondingly that $Q_e^{-1}(1)$ has $2^{g-1}(2^g-1)$ elements.

0.4 *Functoriality*. Let (J, e), (J', e') be two symplectic pairs, and let $\sigma : J \to J'$ be a linear isomorphism compatible with e, e', i.e. verifying

$$e'(\sigma(x), \sigma(y)) = e(x, y) \quad x, y \in J .$$

The isomorphism σ induces a mapping

$$Q(\sigma) : Q(J, e) \to Q(J', e')$$

defined by the formula

$$Q(\sigma)(q) = q \cdot \sigma^{-1} ,$$

and this has the property

$$Q(\sigma)(x+q) = \sigma(x) + Q(\sigma)(q) \quad x \in J, q \in Q(J, e) .$$

Moreover, $Q(\sigma)$ is compatible with the Arf invariant mappings $Q_{e'}$, $Q_{e'}$, in the sense that one has

$$Q_{e'} \cdot Q(\sigma) = Q_e .$$

0.5 *The standard situation*. For a given natural number g (the "genus"), let $J_o = (\mathbf{Z}/2\mathbf{Z})^{2g}$, e_o be defined by the matrix

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

where $0, I$ are respectively the zero, identity $g \times g$ matrix. The datum (J_o, e_o) is a symplectic pair, and is standard in the sense that for a fixed (J, e), giving a symplectic basis for (J, e) amounts to the same thing as giving a linear isomorphism $J_o \simeq J$ compatible with e_o, e. By 0.4, this in turn defines an isomorphism $Q(J_o, e_o) \simeq Q(J, e)$ with the properties stated there.

Going back to the standard situation, there is an obvious identification $Q(J_o, e_o) \simeq (\mathbf{Z}/2\mathbf{Z})^{2g}$, obtained associating with every quadratic form q its values on the canonical basis of J_o. With this identification in mind, the action of J_o on $Q(J_o, e_o)$ defined at the end of 0.2 is the action of $(\mathbf{Z}/2\mathbf{Z})^{2g}$ on itself by translations, and the Arf invariant is given by the mapping Q: $(\varepsilon, \varepsilon') \mapsto \Sigma\, \varepsilon_i \varepsilon_i'$, where $\varepsilon, \varepsilon' \in (\mathbf{Z}/2\mathbf{Z})^g$.

We will use the following notation,

$$\begin{aligned} J_0(g) &= (\mathbf{Z}/2\mathbf{Z})^{2g} \\ S_0(g) &= Q(J_0, e_0) \\ S_0^+(g) &= \{s \in S_0(g) / Q(s) = 0\} \\ S_0^-(g) &= \{s \in S_0(g) / Q(s) = 1\} \end{aligned}$$

§ 1 Theta characteristics

1.1 *On an algebraic curve.* Let C be a non-singular projective algebraic curve over an algebraically closed base field k of characteristic different from 2. The set $S(C)$ of *theta characteristics* on C is the set of isomorphism classes of line bundles L on C whose tensor square is isomorphic to the canonical bundle. If $J_2(C)$ is the group of points of order two in Pic (C), i.e. the multiplicative group of isomorphism classes of line bundles on C whose square is the trivial line bundle $\mathbf{O}_C$, then clearly $J_2(C)$ acts on the set $S(C)$, and this in a simply transitive way. In addition, there is a function

$$Q \colon S(C) \to \mathbf{Z}/2\mathbf{Z}$$

defined by

$$Q(L) = \dim \Gamma(C, L) \quad (2) .$$

The following formula holds, where $x, y \in J_2(C)$, $s \in S(C)$, and we use additive notation both for the group law in $J_2(C)$ and the action of $J_2(C)$ on $S(C)$:

$$Q(s) + Q(x+s) + Q(y+s) + Q(x+y+s) = e(x, y) .$$