§2 Relation with the classical notation

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **22 (1976)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **12.07.2024**

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Throughout this section the base field is **C**.

2.1 *Jacobians.* I recall briefly the data associated with a nonsingular projective curve C. We have two abelian varieties, the Jacobian variety $J(C) = H^{1,0}(C)^*/H_1(C, Z)$ and the Picard variety $P^o(C) = H^{0,1}(C)/\mathrm{H}^1(C, Z)$. From standard dualities it turns out that $P^o(C)$ is naturally isomorphic to the dual Jacobi variety $J(C)^\wedge$, and from Abel's theorem it results that there is in addition a natural isomorphism $P^o(C) \simeq J(C)$. Thus, we have associated with C a principally polarized abelian variety that I will denote henceforth $P^o(C)$, θ_C and will be called the Picard or the Jacobi variety of C according to taste. If we visualize $P^o(C)$ as the group of line bundles on C with Chern class zero, we are led to introduce the family of sets $P^h(C)$, where $P^h(C)$ is the set of isomorphism classes of line bundles with Chern class equal to $h \in \mathbf{Z}$. Each of the sets $P^h(C)$ is a torsor under $P^o(C)$, i.e. is acted on by $P^o(C)$ in a simply transitive way.

There is a natural embedding

$$C \to P^1(C)$$

and it can be proved that this induces an isomorphism of $P^o(C)$-torsors

$$\mathrm{Pic}^\theta P^1(C) \xrightarrow{\sim} P^g(C) \tag{2.1.1}$$

where $\mathrm{Pic}^\theta P^1(C)$ is the set of line bundles P on $P^1(C)$ belonging to θ, and g is the genus of C (see next section 2.2). Observe that $\mathrm{Pic}^\theta P^1(C)$ is properly a $P^o(C)^\wedge$-torsor, but it becomes a $P^o(C)$-torsor through the polarization θ.

2.2 *A simple formalism.* Let X be an abelian variety, P and X-torsor such that the group action $X \times P \to P$ be analytic. Then there are canonical isomorphisms

$$H^i(X, \mathbf{Z}) \simeq H^i(P, \mathbf{Z})$$
$$H^i(X, \mathbf{O}_x) \simeq H^i(P, \mathbf{O}_p)$$

and in particular

$$NS(X) \simeq NS(P)\, \mathrm{Pic}^o(X) \simeq \mathrm{Pic}^o(P)\,.$$

This is because the translations induce the identity both in $H^i(X, \mathbf{Z})$, $H^i(X, \mathbf{O}_x)$ as it may be easily seen. Recall that the Néron-Severi group of X (resp. of P) is the quotient

$$NS(X) = \mathrm{Pic}(X)/\mathrm{Pic}^o(X),$$

or also the kernel of the homomorphism

$$H^2(X, \mathbf{Z}) \to H^2(X, \mathbf{O}_x).$$

Now let $\theta \colon X \to \hat{X}$ be a polarization, θ corresponds naturally to an element $\theta \in NS(X)$, and the set $\mathrm{Pic}^\theta(X)$ of isomorphism classes of line bundles on X belonging to θ is the coset of $\hat{X}$ in $\mathrm{Pic}(X)$ corresponding to θ (cf. for example, Mumford, Abelian Varieties). Thus, $\mathrm{Pic}^\theta(P)$ is well defined too, since $NS(P)$ and $NS(X)$ identify.

Starting from (X, θ) and P we have the following situation. The set $\mathrm{Pic}^\theta(P)$ is a torsor over $\mathrm{Pic}^o(P)$, but $\mathrm{Pic}^o(P)$ identifies naturally with $\hat{X}$, thus $\mathrm{Pic}^\theta(P)$ is an $\hat{X}$-torsor. The following formula makes explicit this $\hat{X}$-torsor as tensor product (the natural operation between torsors over a fixed abelian group) of two other $\hat{X}$-torsors, $\mathrm{Pic}^\theta X$ and the $\hat{X}$-torsor $P \otimes_X \hat{X}$ obtained from P through the extension of scalars $\theta \colon X \to \hat{X}$.

$$\mathrm{Pic}^\theta(P) \simeq \mathrm{Pic}^\theta(X) \otimes (P \otimes_X \hat{X}) \tag{2.2.1}$$

To have this natural isomorphism it is enough to define an X-equivariant pairing $\mathrm{Pic}^\theta(X) \times P \to \mathrm{Pic}^\theta(P)$ and this is the obvious one: if $L \in \mathrm{Pic}^\theta(X)$, $p \in P$ and if $t_p \colon X \to P$ is the isomorphism $t_p(x) = p + x$, then the pairing associates with (L, p) the line bundle $(t_p)_*(L)$.

This isomorphism will be used in the next section.

2.3 *Relation between 1.1, 1.2.* Let C be a nonsingular projective algebraic curve, $(P^o(C), \theta_C)$ its Picard variety with its principal polarization. Then, the definitions of theta characteristics of 1.1, 1.2 applied respectively to C, $(P^o(C), \theta_C)$ yield objects that identify naturally. Indeed, if follows from (2.1.1) and (2.2.1) that for any $h \in Z$ there is a natural isomorphism of $P^o(C)$-torsors.

$$\mathrm{Pic}^\theta(P^h(C)) \simeq P^{h+g-1}(C),$$

where g is the genus of C. In particular, we have isomorphisms

$$\mathrm{Pic}^\theta(P^o(C)) \simeq P^{g-1}(C)$$
$$\mathrm{Pic}^{2\theta}(P^o(C)) \simeq P^{2g-2}(C).$$

In the last one it is easily seen that the canonical bundle corresponds to the unique totally symmetric bundle in $\mathrm{Pic}^{2\theta} P^o(C)$. As the symmetric bundles in $\mathrm{Pic}^{\theta}(P^{o}(C))$ are exactly the square roots of this totally symmetric line bundle, it follows that $S(C)$, $S(P^{o}(C), \theta_{C})$ identify naturally. Moreover, this identification is compatible with their structures of $J_2(C)$-torsors and with the maps $Q: S(C) \to \mathbf{Z}/2\mathbf{Z}$, $Q: S(P^{o}(C), \theta_{C}) \to \mathbf{Z}/2\mathbf{Z}$. This last point follows easily from proposition 2 in § 2 of Mumford [4] and from the theorem of Riemann (see Fay [2], theorem 1.1) stating that for a line bundle $L \in P^{g-1}(C)$, the dimension of $\Gamma(C, L)$ equals the multiplicity of the theta divisor at the point L. (In fact, observe that the theta divisor as an element of $\mathrm{Pic}^{\theta}(P^{g-1}(C))$ corresponds to the canonical bundle on C under the isomorphism $\mathrm{Pic}^{\theta}(P^{g-1}(C)) \approx P^{2g-2}(C)$.

2.4 *Theta functions.* Let (X, θ) be a principally polarized abelian variety. There is a canonical isomorphism

$$X \simeq H^{1,0}(X)^{*}/H_1(X, \mathbf{Z})$$

and the principal polarization corresponds to a nondegenerate alternate bilinear pairing

$$\theta: H_1(X, \mathbf{Z}) \, x \, H_1(X, \mathbf{Z}) \to \mathbf{Z}\,.$$

Let $x_1, \ldots, x_g, x_1', \ldots, x_g'$ be a symplectic basis for θ on $H_1(X, Z)$; then the images of $x_1', \ldots, x_g'$ in $H^{1,0}(X)^{*}$ constitute a basis for this $\mathbf{C}$-vector space, and let $w_1, \ldots, w_g$ be its dual basis for $H^{1,0}(X)$. In other words,

$$\int_{x_i'} w_j = \delta_{ij}\,.$$

Then the matrix $\tau = (\tau_{ij})$ defined by

$$\tau_{ij} = \int_{x_i} w_j$$

belongs to the Siegel upper-half space of degree g, i.e. τ is symmetric and $Im(\tau)$ is positive definite. The choice of the symplectic basis sets an identification

$$X \simeq \mathbf{C}^{g}/(\tau\mathbf{Z}^{g} \oplus \mathbf{Z}^{g})\,.$$

We may now consider the classical theta functions (Igusa [3])

$$\theta_{mm^*}(\tau, z) = \sum_{\xi \in \mathbf{Z}^g} \mathbf{e}\left[\tfrac{1}{2}(\zeta + m)\,\tau^{t}(\zeta + m) + (\zeta + m)^{t}(z + m^{*})\right].$$

By the properties of these theta functions and through the preceding identification, each $\theta_{mm^*}(\tau, -)$ defines a line bundle on X, and indeed an element of $\mathrm{Pic}^{\theta}(X)$ that is independent of $(m, m^{*}) \in \mathbf{R}^{2g} \bmod \mathbf{Z}^{2g}$. In this way we get a bijection

$$\mathrm{Pic}^{\theta}(X) \simeq \mathbf{R}^{2g}/\mathbf{Z}^{2g}\,.$$

It follows from formula (θ. 1) in p. 49 of Igusa [3] that the subset of $\mathrm{Pic}^{\theta}(X)$ defined by the symmetric line bundles on X corresponds to the image in $\mathbf{R}^{2g}/\mathbf{Z}^{2g}$ of $\frac{1}{2}\,\mathbf{Z}^{2g}$.

We finally see that the symplectic basis on $H_1(X, Z)$ defines an identification

$$S(X, \theta) \simeq (\mathbf{Z}/2\mathbf{Z})^{2g}.$$

It is easy to see that this identification depends only on the symplectic basis induced on

$$H_1(X, \mathbf{Z})/2H_1(X, \mathbf{Z}) \simeq H_1(X, \mathbf{Z}/2\mathbf{Z}),$$

and that it is compatible with the identification

$$\hat{X}_2 \simeq H_1(X, \mathbf{Z}/2\mathbf{Z}) \simeq (\mathbf{Z}/2\mathbf{Z})^{2g}$$

that the later basis defines and with the respective action of $\hat{X}_2$ on $S(X, \theta)$ and of $(\mathbf{Z}/2\mathbf{Z})^{2g}$ on itself by translations.

2.5 *Summing up.* If C is a nonsingular projective algebraic curve of genus g, there are two equivalent ways of defining the set of theta characteristics, either directly as in 1.1, or through its Picard variety as in 1.2. The set of theta characteristic is endowed with a simply transitive action of the group $J_2(C)$ and with a function $Q\colon S(C) \to \mathbf{Z}/2\mathbf{Z}$ closely related to the intersection pairing e on $J_2(C)$. Also, we know that $Q^{-1}(0)$ has $2^{g-1}(2^g+1)$ elements and $Q^{-1}(1)$ has $2^{g-1}(2^g-1)$ elements. Indeed, there is a third way of defining the set of theta characteristics, namely as the set $Q\big(J_2(C), e\big)$ of all quadratic forms g on $J_2(C)$ whose associated bilinear form is e; we saw in § 0 that on this set there is a structure of the same type as in $S(C)$, $S(X, \theta)$, and in fact $S(X, \theta)$ is clearly isomorphic with $Q(\hat{X}_2, e) \simeq Q(X_2, e)$.

Now if we choose a symplectic basis $x_1, \ldots, x_g, x_1', \ldots, x_g'$ for $J_2(C)$, the set $S(C)$ identifies with $(\mathbf{Z}/2\mathbf{Z})^{2g}$. In particular, $0 \in (\mathbf{Z}/2\mathbf{Z})^{2g}$ defines a "base" theta characteristic. In terms of quadratic forms, this identification corresponds to the one discussed in 0.5, in particular the base theta characteristic is even (i.e. belongs to $Q^{-1}(0)$) and it corresponds to the quadratic form q_o defined by $q_o(x_i) = q_o(x_i') = 0$. for $i = 1, \ldots, g$. Looking at $S(C)$ as a subset of $P^{g-1}(C)$, the base theta characteristic is nothing else that the Riemann constant Δ in the non-intrinsic version of the Riemann theorem referred to at the end of 2.3. (See theorem 1.1 in Fay [2] and its corollary 1.5).