§3 Some special cases

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§ 3 SOME SPECIAL CASES

I present here some examples in order to motivate the general discussion in Part II. Proofs of most assertions are omitted and they may be found in or follow easily from Part II. The base field is $\mathbf{C}$ to simplify things.

3.1 *Genus two.* Let C be of genus two, and let P_C be the projective space of hyperplanes in $H^{1,0}(C)$. Then P_C is a projective line, and the natural map $C \to P_C$ presents C as a 2-sheeted covering of P_C ramified over a subset $R_C \subset P_C$ with $|R_C| = 6$. From the Riemann-Roch theorem it may be proved that the line bundles L in $S(C)$ with $Q(L) = 1$, i.e. the odd theta characteristics, are those represented by effective divisors, and from here it follows easily that the set $S(C)$ of odd theta characteristics identifies naturally with R_C. If s_1, s_2, s_3 are three different elements of $S^-(C)$ represented by line bundles L_1, L_2, L_3, it is also easily proved that $L_1 \oplus L_2 \oplus L_3^{-1}$ is even. From this, and from II 2.4 it follows that there is a natural group isomorphism

$$\mathrm{Sp}\left(H_1(C, \mathbf{Z}/2\mathbf{Z})\right) \simeq \mathrm{Aut}(R_C).$$

It follows also from *loc. cit.* that it amounts to the same thing to give a symplectic basis for $H_1(C, \mathbf{Z}/2\mathbf{Z})$ or to give a bijection $S_0^-(2) \simeq R_C$, where $S_0^-(2)$ is the fixed 6-elements set defined in 0.5.

I will discuss $S^+(C)$ in a more general setting:

3.2 *Even genus, hyperelliptic case.* Let C be hyperelliptic. Then there is a projective line P_C and a map $C \to P_C$ defined up to unique isomorphisms such that $C \to P_C$ is a 2-sheeted covering. If R_C is the ramification locus, $|R_C| = 2g + 2$, and R_C identifies naturally with the set of Weierstrass points of C.

The group $H_1(C, \mathbf{Z}/2\mathbf{Z})$ can be reconstructed starting from R_C in the following way. If $\pi = \{\pi', \pi''\}$ is any partition of R_C into two even-order subsets, L_π is the line bundle defined by the divisor $\sum_{P' \in \pi_1'} P - \sum_{P \in \pi_2''} P$ where $|\pi_1'| = |\pi_2'|$ and $\{\pi_1', \pi_2'\}$ partition π'. It is clear that L_π is of order two, thus defining an element of $H_1(C, \mathbf{Z}/2\mathbf{Z})$. In this manner one gets a group isomorphism

$$P_2^+(R_C) \xrightarrow{\sim} H_1(C, \mathbf{Z}/2\mathbf{Z})$$

where the group $P_2^+(R_C)$ is defined in II 3.5. It is easily verified that this isomorphism is compatible with the intersection pairing on H_1 and with the alternated bilinear form introduced in *loc. cit.*

All the preceding was valid for any genus g. Now if g is even, it follows from II 3.6 and II 1.4 that we have an isomorphism

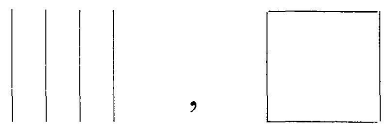
$$P_2^-(R_C) \xrightarrow{\sim} S(C)$$

compatible with the structures involved (i.e. an isomorphism of symplectic torsors, cf. II 1.1). The results of II, § 3 may thus be applied to the study of $S(C)$.

Observe that if g is odd, there is a natural theta characteristic; namely, the line bundle of the divisor $(g-1)P$ is independent of the Weierstrass point P (compare II 3.6b)).

3.3 *Genus three.* Two cases arise for C of genus three:

3.3.1 *C hyperelliptic.* Then there is the 2-sheeted covering $C \to P_C$ ramified over R_C with $|R_C| = 8$. It is seen in this case, as in 3.1, that there is a natural identification between $S^-(C)$ and the set of subsets of R_C consisting of exactly two elements. It is convenient to visualize the elements of $S^-(C)$ as segments joining the points of R_C, these being distributed on a plane in an arbitrary way. Then, if s_1, s_2, s_3, s_4 are four different elements of $S^-(C)$, $s_1 - s_2 = s_3 - s_4$ iff the segments corresponding to them produce one of the following configurations



From II 2.7 it follows that there is a canonical isomorphism between the group $Sp(H_1(C, \mathbf{Z}/2\mathbf{Z}))$ and the group of permutations of the set $S^-(C)$ that preserve the "geometry" defined by these quadruples. Two comments are in order:

a) Although the permutation group $\operatorname{Aut}(R_C)$ is clearly a subgroup of the automorphism group of the "geometry", not every such automorphism arises from a permutation of R_C.

b) The automorphisms of the geometry do not preserve the type of the configuration, they may send one quadruple of the first type drawn above into the other. However in a continuous family of *hyperelliptic* curves of genus 3, each of the two configurations will be preserved as the curve is deformed.

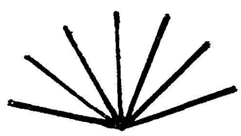
3.3.2 *C non hyperelliptic.* Let $Q_C = \mathbf{P}(H^{1,0}(C))$ be the projective space of hyperplanes in $H^{1,0}(C)$. Then Q_C is a projective plane and the natural map $C \to Q_C$ is an immersion. The degree of C in Q_C is the degree of the canonical bundle, i.e. 4 and C is thus a *nonsingular plane quartic.* It is again a simple exercise to prove that the odd theta characteristics on C correspond to the set of lines in Q_C that are bitangents to C. Thus, if B_C is the set of bitangents to C in Q_C, there is a natural identification

$$B_C \simeq S^-(C)\,.$$

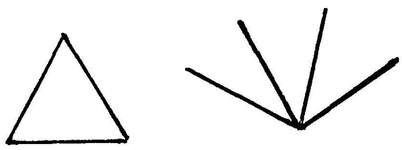
The theme of the 28 bitangents to a nonsingular plane quartic ($28 = 2^{3-1}(2^3-1)$) is a classic one in geometry, see for instance Weber [6], chapter 12. A triple (s_1, s_2, s_3) of bitangents is called *syzygetic* (resp. *azygetic*) if their six points of contact with C lie (resp. do not lie) in a conic. A triple is syzygetic iff $L_4 = L_1 \otimes L_2 \otimes L_3^{-1}$ is an odd characteristic, where L_1, L_2, L_3 are the line bundles corresponding to s_1, s_2, s_3. When this happens, the two points of contact of the bitangent s_4 corresponding to L_4, together with the preceding six, make up the full $8 = 2 \times 4$ common points of the conic with the quartic.

An *Aronhold system* of bitangents (Weber [6]) is a set of seven bitangents such that any different three of them constitute an azygetic triple. The Aronhold systems are exactly the basis for the "geometry" in $S^-(C)$ defined by the syzygetic triples (in the sense of II 4.3). It follows from II 4.4 that the set of Aronhold systems is a torsor over the symplectic group $Sp(H_1(C, \mathbf{Z}/2\mathbf{Z}))$, in particular that they have the same number of elements.

As any two "geometries" with the same genus are isomorphic (II 1.4), one can also speak of Aronhold systems in the hyperelliptic case. It turns out that they correspond to the following configurations



and



There are 1,451,520 of them as it is "immediately" checked. Again, it will be observed that the automorphisms of the geometry do not preserve the type of the configuration.