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Kapitel: §1 Symplectic torsors

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II. THE ABSTRACT THEORY OF CHARACTERISTICS

§ 1 Symplectic torsors

1.1 *Definitions.* Recall that, if Γ is a group, a Γ-torsor (or torsor over Γ) is a non-void set endowed with a simply transitive action of Γ on it. Let (J, e) be a symplectic pair, a *symplectic torsor* over (J, e) is a pair (S, Q) of a J-torsor S and a mapping $Q: S \to \mathbf{Z}/2\mathbf{Z}$ having the property

$$(1.1.1) \qquad Q(s) + Q(x+s) + Q(y+s) + Q(x+y+s) = e(x, y)$$

where $s \in S$, $x, y \in J$. It is clearly equivalent to ask this property for a fixed $s \in S$ or for all $s \in S$, and it may be thought of as meaning that Q "is a quadratic form." Indeed, any $s \in S$ sets an identification $J \xrightarrow{\sim} S$ $(x \mapsto x+s)$, and through this identification Q becomes the map $x \mapsto Q(x+s)$. The above property means that the map $q_s: J \to \mathbf{Z}/2\mathbf{Z}$ defined by

$$(1.1.2) \qquad q_s(x) = Q(x+s) + Q(s)$$

is a quadratic form whose associated bilinear form is e. According to 0.4, two possibilities may and do arise for Q: either $Q^{-1}(0)$ has $2^{g-1}(2^g+1)$ or $2^{g-1}(2^g-1)$ elements, where $g = \dim J/2$ will be called the *genus* of (S, Q). In the first case, (S, Q) will be said to be *even*, *odd* in the second. In what follows, *all symplectic torsors will be even* unless otherwise stated. This because the symplectic torsors that will appear most often will be even and because of the following simple construction. If (S, Q) is an even (resp. odd) symplectic torsor over (J, e), and $\bar{Q}$ is defined by $\bar{Q}(s) = Q(s) + 1$, then $(S, \bar{Q})$ is an odd (resp. even) symplectic torsor over (J, e).

For a given (S, Q) the following notation will be used

$$S^+ = Q^{-1}(0) \quad S^- = Q^{-1}(1).$$

The elements of S will be often called *characteristics*, those in S^+ are *positive*, those in S^- are *negative*.

1.2 *Morphisms.* Let (S, Q), (S', Q') be symplectic torsors respectively over (J, e), (J', e'). For any map $f: S \to S'$ we define a map $\sigma_f: J \times S \to J'$ by the property

$$f(x+s) = \sigma_f(x, s) + f(s);$$

this can be done because S' is a J'-torsor. Now, the following cocycle-type property for σ_f is immediately checked, where $x, y \in J$, $s \in S$

$$\sigma_f(x+y+s) = \sigma_f(x, y+s) + \sigma_f(y, s),$$

and from it one infers the equivalence of the following statements:

(i) For any $s, s' \in S$, $x \in J$

$$\sigma_f(x, s) = \sigma_f(x, s').$$

(ii) For some $s \in S$, any $x, y \in J$

$$\sigma_f(x+y, s) = \sigma_f(x, s) + \sigma_f(y, s)$$

(iii) For any $s \in S$, $x, y \in J$

$$\sigma_f(x+y, s) = \sigma_f(x, s) + \sigma_f(y, s).$$

So, when these statements hold, one gets a group homomorphism $\sigma_f : J \to J'$ and has $f(x+s) = \sigma_f(x) + f(s)$.

An *isomorphism* of (S, Q) onto (S', Q') is a bijection $f : S \to S'$ verifying statements (i) to (iii) above, and also the condition

$$Q' \circ f = Q.$$

It is clear in this case that $\sigma_f : J \to J'$ is an isomorphism compatible with e, e'. The group of automorphisms of (S, Q) will be denoted $Sp(S, Q)$, so the mapping $f \to \sigma_f$ is a group homomorphism $Sp(S, Q) \to Sp(J, e)$.

1.3 *An example*. For any given (J, e) there is a canonical example of an even symplectic torsor, namely $(Q(J, e), Q_e)$. The J-torsor $Q(J, e)$ was introduced in 0.2, the map Q_e in 0.3 where it was also remarked that it has property (1.1.1) and that $Q_e^{-1}(0)$ has $2^{g-1}(2^g+1)$ elements.

If (J, e), (J', e') are two symplectic pairs, and if $\sigma : J \to J'$ is a linear isomorphism compatible with e, e', a map $Q(\sigma) : Q(J, e) \to Q(J', e')$ was defined in 0.4, where it was shown that it is an isomorphism of symplectic torsors. Clearly $Q(\sigma)$ is canonical in any conceivable way.

Indeed, if one still dares in these days to use the language of category theory, what I just did was to define a functor from the category of symplectic pairs to the category of even symplectic torsors (morphisms = isomorphisms, in both cases). In section 1.4 we will see that this is an equivalence of categories.

1.4 *Uniqueness of symplectic torsors.* It will be shown here, that for a given symplectic pair (J, e) there is essentially only one symplectic torsor over it. Let (S, Q) be such an object; then there is a map

$$f_s \colon S \to Q(J, e),$$

defined by the rule $s \mapsto q_s$, where q_s was defined in (1.1.2). Let us prove that f_s is an isomorphism of symplectic torsors inducing the identity $id_J \colon J \to J$. The formula

$$q_{x+s}(y) = (x + q_s)(y)$$

is a mere restatement of condition (1.1.1), and the formula

$$Q_e \circ f_s = Q$$

follows from the fact that (S, Q) is even and from the meaning of the Arf invariant recalled in 0.3.

The isomorphisms f_s are canonical, in the following sense. If (S, Q), (S', Q') are symplectic torsors over (J, e), (J', e'), $f \colon S \to S'$ is an isomorphism of symplectic torsors inducing an isomorphism $\sigma \colon J \to J'$, then the following square commutes

$$\begin{array}{ccc} S & \xrightarrow{\ f\ } & S' \\ {\scriptstyle f_s}\downarrow & & \downarrow{\scriptstyle f_{s'}} \\ Q(J, e) & \xrightarrow{\ Q(\sigma)\ } & Q(J', e') \end{array}$$

Recalling the definitions, one has to check for $s \in S$, $x \in J$ that

$$Q(\sigma(x) + f(s)) + Q(f(s)) = Q(x + s) + Q(s)$$

which is immediate from the definition of isomorphism in 1.2.

It comes out of this that for any isomorphism $\sigma \colon J \to J'$ there exists one and only one isomorphism $f \colon S \to S'$ inducing it. In particular, the group homomorphism at the end of 1.2.

$$Sp(S, Q) \to Sp(J, e)$$

is an isomorphism. A useful application of this is the following: If by some unspecified means one is able to construct two symplectic torsors over a pair (J, e), there is a unique isomorphism between them inducing the identity of J.

1.5 *Some notation.* a) Let J be a vector space over $\mathbf{Z}/2\mathbf{Z}$, S a J-torsor. Let's put

$$E(S) = J \amalg S$$

the disjoint union of J, S; on this set there is a structure of vector space over $\mathbf{Z}/2\mathbf{Z}$. In fact there is an exact sequence

$$0 \to J \to E(S) \to \mathbf{Z}/2\mathbf{Z} \to 0$$

where J is sent identically onto itself, and the inverse image of 0 (resp. 1) in $E(S)$ is J (resp. S). The addition law in $E(S)$ reduces to the given one on J when both elements are in J, is the action of J on S when one element is in J and the other in S, and finally $s + s'$ (for $s, s' \in S$) is the unique element $x \in J$ such that $x + s = s'$ (or equivalently $x + s' = s$).

b) Given the standard pair (J_o, e_o), as in 0.5. I will write $S_o = Q(J_o, e_o)$, $Q_o = Q_{e_o}$. Both J_o, S_o identify to $(\mathbf{Z}/2\mathbf{Z})^{2g}$, but the following notations will be used in compliance with tradition, where $u_1, \ldots, u_{2g}$ is the canonical basis. An element of the form

$$\sum_{i=1}^{g} (\varepsilon_i u_i + \varepsilon_i' u_{i+g})$$

will be written $\begin{pmatrix} \varepsilon \\ \varepsilon' \end{pmatrix}$ or $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$ whether it is seen in J_o or S_o respectively, where ε, ε' are row vectors. In particular, the addition law in $E(S_o)$ is the following:

$$\begin{pmatrix} \varepsilon \\ \varepsilon' \end{pmatrix} + \begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} \varepsilon + \eta \\ \varepsilon' + \eta' \end{pmatrix}$$

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§ 2 FINITE GEOMETRIES ON SETS OF CHARACTERISTICS

2.0 Let's fix for paragraph § 2 a symplectic torsor (S, Q) over a symplectic pair (J, e) of genus g. The letter Σ will stand for either the set S^+ of S^-, its cardinality is $2^{g-1}(2^g \pm 1)$ (recall that according to 1.1 we assume