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A DIRECTIONAL CLUSTER SET EXAMPLE

by C. L. Belna, M. J. Evans and P. D. Humke

§0. Introduction

Let f be a mapping from the open upper half plane H into the Riemann sphere W. For each point x on the real line R, let $C(f, x)$ and $C(f, x, \theta)$ denote respectively the total cluster set of f at x and the cluster set of f at x in the direction θ $(0<\theta<\pi)$; then let $\Theta(x)$ denote the set of directions θ for which $C(f, x, \theta) = C(f, x)$. E. F. Collingwood [3, Theorem 2 combined with Theorem 3] established the following result.

Theorem C. *Let $f: H \to W$ be continuous. Then the set $\Theta(x)$ is residual at each point x of a residual subset S of R.*

A. M. Bruckner and Casper Goffman [2, p. 510] raised the question as to whether or not there exists a residual set of directions Θ such that $\Theta \subset \Theta(x)$ for each $x \in S$. Here we prove

Theorem 1. *There exists a continuous $f: H \to W$ such that $\bigcap_{x \in Q} \Theta(x)$ is a first category set of directions for each residual subset Q of R.*

To construct this function (§3), we use certain sets of J.-P. Kahane [5] as building blocks [1]). Two important properties of these sets are established in §2, and the necessary technical preliminaries are presented in §1. Finally, in §4 we present an example concerning essential directional cluster sets.

§1. (1/2)-trapezoids and their four descendants

By a (1/2)-*trapezoid* we mean any closed trapezoid T having bases L and L' which lie respectively on the lines $y = 0$ and $y = 1$ and for which $|L| = 2|L'|$. (Here and throughout this paper, we use $|\tau|$ to denote the length of the line segment τ.) For each (1/2)-trapezoid T, we set

$$T(1/2) = \{ z \in T : \operatorname{Im}(z) = 1/2 \} .$$

[1]) The authors wish to thank Professor John R. Kinney for bringing this paper of J.-P. Kahane to their attention.

For real numbers p and p', let $\tau_{pp'}$ denote the line segment joining the points $(p, 0)$ and $(p', 1)$. Then the *projection of the segment* $\tau_{pp'}$ is given by

$$\operatorname{proj}[\tau_{pp'}] = p' - p,$$

and the *projection set of the* (1/2)-*trapezoid* T is given by

$$\operatorname{proj}[T] = \{\operatorname{proj}[\tau_{pp'}] : (p, 0) \in L \quad \text{and} \quad (p', 1) \in L'\}.$$

For the remainder of this section, suppose T is a (1/2)-trapezoid with bases L and L', and let L_1 and L_2 (resp., L_1' and L_2') denote the two line segments that remain when the open middle half of L (resp., L') is removed with L_1 (resp., L_1') lying to the left of L_2 (resp., L_2'). Then the *four descendants* of T are the (1/2)-trapezoids T_1, T_2, T_3, and T_4 having respective bases L_1 and L_1', L_1 and L_2', L_2 and L_1', and L_2 and L_2'.

Now let a and a' denote the respective x-coordinates of the left end-points of L and L', and set $l = |L|$. Then the following list of facts concerning T and its descendants can easily be established:

(A) If $\hat{a} = (a+a')/2$, then

$$T(1/2) = \{(x, 1/2) : \hat{a} \leqslant x \leqslant \hat{a} + 3l/4\}$$

and, for each $k = 1, 2, 3$, and 4, we have

$$T_k(1/2) = \{(x, 1/2) : \hat{a} + 3(k-1)\,l/16 \leqslant x \leqslant \hat{a} + 3kl/16\}.$$

That is, the segments $T_k(1/2)$ $(k=1, 2, 3, 4)$ partition the segment $T(1/2)$ into four equal subsegments.

(B) If $\hat{v} = a' - a$, then

$$\operatorname{proj}[T] = \{v : \hat{v} - l \leqslant v \leqslant \hat{v} + l/2\},$$

$$\operatorname{proj}[T_3] = \{v : \hat{v} - l \leqslant v \leqslant \hat{v} - 5l/8\},$$

$$\operatorname{proj}[T_4] = \{v : \hat{v} - 5l/8 \leqslant v \leqslant \hat{v} - 2l/8\},$$

$$\operatorname{proj}[T_1] = \{v : \hat{v} - 2l/8 \leqslant v \leqslant \hat{v} + l/8\},$$

and

$$\operatorname{proj}[T_2] = \{v : \hat{v} + l/8 \leqslant v \leqslant \hat{v} + 4l/8\}.$$

That is, the intervals proj $[T_k]$ $(k=1, 2, 3, 4)$ partition the interval proj $[T]$ into four equal subintervals.

§2. Kahane's set K_ξ.

Let ξ be a real number, and let T_ξ be the (1/2)-trapezoid with bases

$$L = \{(x, 0): 0 \leqslant x \leqslant 1\} \quad \text{and} \quad L' = \{(x', 1): \xi \leqslant x' \leqslant \xi + 1/2\}.$$

Let T_{11}, T_{12}, T_{13}, and T_{14} be the four descendants of T_ξ labeled in such a way that the points

$$\alpha_{1k} = \min\{x: (x, 1/2) \in T_{1k}\}$$

satisfy $\alpha_{1k} < \alpha_{1(k+1)}$ for $k = 1, 2,$ and 3. Then let T_{2k} ($k = 1, 2, \ldots, 4^2$) be the 4^2 descendants of the trapezoids T_{1k} ($k = 1, 2, 3, 4$) labeled in such a way that the points

$$\alpha_{2k} = \min\{x: (x, 1/2) \in T_{2k}\}$$

satisfy $\alpha_{2k} < \alpha_{2(k+1)}$ for $k = 1, 2, \ldots, 4^2 - 1$. Continuing inductively we arrive at the collection of (1/2)-trapezoids

$$T_{nk} \quad \text{for} \quad n = 1, 2, \ldots \quad \text{and} \quad k = 1, 2, \ldots, 4^n$$

where the T_{nk} ($k = 1, 2, \ldots, 4^n$) are the descendants of the $T_{(n-1)k}$ ($k = 1, 2, \ldots, 4^{n-1}$) and are labeled in such a way that the points

$$\alpha_{nk} = \min\{x: (x, 1/2) \in T_{nk}\}$$

satisfy $\alpha_{nk} < \alpha_{n(k+1)}$ for $k = 1, 2, \ldots, 4^n - 1$.

If for each n we set

$$\alpha_{n(4^n+1)} = \max\{x: (x, 1/2) \in T_{n4^n}\},$$

then it follows from (A) in §1 that both

$$T_{nk}(1/2) = \{(x, 1/2): \alpha_{nk} \leqslant x \leqslant \alpha_{n(k+1)}\}$$

for each $k = 1, 2, \ldots, 4^n$, and for each n

$$T_\xi(1/2) = \bigcup_{k=1}^{4^n} T_{nk}(1/2).$$

Now, for each n, let $K_n = \bigcup_{k=1}^{4^n} T_{nk}$ and define the set

$$K_\xi = \bigcap_{n=1}^{\infty} K_n.$$

The following properties of the set K_ξ were established by Kahane [5]:

(I) K_ξ *is a compact set of 2-dimensional measure zero.*

(II) *Let any segment* τ *that extends from one base of* T_ξ *to the other be called "admissible". Then the following two properties hold:*

(i) *There are at most two admissible line segments in* K_ξ *that pass through any given point* $z \in T_\xi(1/2)$; *in fact, precisely one such segment passes through each* $z \in T_\xi(1/2)$ *that is not one of the* α_{nk}.

(ii) *For each* $\lambda \in [\xi-1, \xi+1/2] = \mathrm{proj}\,[T_\xi]$, *there exists at least one point* $z \in T_\xi(1/2)$ *through which there passes an admissible line segment* τ *in* K_ξ *with* $\mathrm{proj}\,[\tau] = \lambda$; *in fact, for all but countably many such* λ, *there exists only one such* z.

We now introduce a set-valued function Λ defined on the subsets A of $T_\xi(1/2)$ as follows:

$$\Lambda(A) = \{\, \mathrm{proj}\,[\tau] : \tau \subset K_\xi \text{ is an admissible segment with } A \cap \tau \neq \phi \,\}.$$

Then K_ξ has the two additional properties:

(III) $\Lambda(A)$ *is nowhere dense whenever* A *is.*

(IV) $\Lambda(A)$ *is of linear (Lebesgue) measure zero whenever* A *is.*

Proof of (III). Let A be a nowhere dense subset of $T_\xi(1/2)$, and suppose $\Lambda(A)$ is dense on some subinterval I of the interval $\mathrm{proj}\,[T_\xi]$. Then, according to (B) in §1, there exist integers n and k such that $\Lambda(A)$ is dense on $\mathrm{proj}\,[T_{nk}]$. Since A is nowhere dense, there exist integers n' and k' such that $T_{n'k'}(1/2) \subset T_{nk}(1/2)$ and $A \cap T_{n'k'} = \phi$. By (B) of §1, it follows that

$$\Lambda(A) \cap \mathrm{int}(\mathrm{proj}\,[T_{n'k'}]) = \phi$$

(int $\equiv$ interior). This contradicts the fact that $\Lambda(A)$ is dense on $\mathrm{proj}\,[T_{nk}]$, and property (III) is proved.

For the remainder of this paper, we use $\mu(A)$ and $\mu^*(A)$ to denote respectively the Lebesgue measure and the Lebesgue outer measure of the linear set A.

Proof of (IV). Let $A \subset T_\xi(1/2)$ and suppose $\mu(A) = 0$. If we set

$$\hat{A} = A - \{\, \alpha_{nk} : n = 1, 2, \ldots;\ k = 1, 2, \ldots, 4^n + 1 \,\},$$

then in view of II (i) it suffices to show that the corresponding set $\Lambda(\hat{A})$ has linear measure zero.

Let $\mathscr{I}$ be the collection of all closed intervals on $T_\xi(1/2)$ of the form $[\alpha_{nk}, \alpha_{n(k+1)}]$. Then it is easy to verify that

$$\mu(\hat{A}) = \inf \sum_{j=1}^{\infty} \mu(I_j),$$

where the inf is taken over all sequences $\{I_j\}$ of intervals in $\mathscr{I}$ covering $\hat{A}$.

Let $\varepsilon > 0$. Then there is a sequence $\{I_j\}$ of intervals in $\mathscr{I}$ that cover $\hat{A}$ such that

$$\sum_{j=1}^{\infty} \mu(I_j) < \varepsilon .$$

For each index j, there exist integers n_j and k_j such that

$$I_j = T_{n_j k_j}(1/2);$$

hence, in view of (B) in §1, we have

$$\Lambda(\hat{A}) \subset \bigcup_{j=1}^{\infty} \operatorname{proj}[T_{n_j k_j}] .$$

Furthermore, combining (A) and (B) of §1, we obtain

$$\mu(\operatorname{proj}[T_{n_j k_j}]) = 2\mu(I_j) \quad (j=1,2,\ldots) .$$

Therefore,

$$\mu^*(\Lambda(\hat{A})) \leqslant \sum_{j=1}^{\infty} \mu(\operatorname{proj}[T_{n_j k_j}]) = 2\sum_{j=1}^{\infty} \mu(I_j) < 2\varepsilon ,$$

and property (IV) is proved.

§3. Proof of Theorem 1

For each integer n, let $\xi_n = 1 + 3n/2$ and set

$$\Delta_n = K_{\xi_n} \cap \{(x, y) : \xi_n/2 \leqslant x \leqslant \xi_n/2 + 3/4 \quad \text{and} \quad 1/2 < y \leqslant 1\} .$$

Then set

$$\Delta_n^* = \{ z - (1+i)/2 : z \in \Delta_n \} \quad (i = \sqrt{-1}) ,$$

and define the set

$$\Delta = \cup \{\Delta_n^* : n = 0, \pm 1, \pm 2, \ldots\} .$$

Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of points in $H - \Delta$ whose derived set is R. Define the function f_0 on $\Delta \cup \{z_n\}_{n=1}^{\infty}$ by

$$f_0(z) = \begin{cases} 0 & \text{for} \quad z \in \Delta \\ 1 & \text{for} \quad z = z_n (n = 1, 2, \ldots) . \end{cases}$$

Then let f be a continuous extension of f_0 to all of H. We now show that f is the desired function.

Let Q be a residual subset of R. Then, for each integer n, the set $Q_n = Q \cap [3n/4, 3(n+1)/4]$ is a residual subset of the closed interval $[3n/4, 3(n+1)/4]$. As a consequence of (III) for $\xi = \xi_n$, there exists a residual set of directions $\Theta_n \subset \{ \theta : 3n/2 \leqslant \cot \theta \leqslant 3(n+1)/2 \}$ such that, for each $\theta \in \Theta_n$, there exists a segment in Δ emanating from a point of Q_n and having the direction θ. Therefore, the set $\bigcap_{x \in Q} \Theta(x)$ is of the first category on the set $\{ \theta : 3n/2 \leqslant \cot \theta \leqslant 3(n+1)/2 \}$ for each integer n, and the theorem is proved.

§4. An Essential Cluster Set Example

If f is a measurable function from H to W, then the *essential cluster set* $C_e(f, x)$ *of* f *at* x is defined as the set of all values $w \in W$ for which the upper density of $f^{-1}(U)$ at x is positive for every open set U containing w; the *essential cluster set* $C_e(f, x, \theta)$ *of* f *at* x *in the direction* θ is the set of all values $w \in W$ for which the upper density of $f^{-1}(U)$ along the ray at x having direction θ is positive for every open set U containing w. As a supplement to a result of Casper Goffman and W. T. Sledd [4, Theorem 2], the present authors [1] proved the following result concerning the set

$$\Theta^*(x) = \{ \theta : C_e(f, x) \subset C_e(f, x, \theta) \} \quad (x \in R) .$$

Theorem B.E.H. *If* $f : H \to W$ *is measurable, then* $\mu(\Theta^*(x)) = \pi$ *for almost every and nearly every* $x \in R$; *furthermore, if* f *is continuous, then* $\Theta^*(x)$ *is residual for almost every and nearly every* $x \in R$.

Again, a natural question to ask is whether or not, for a given function f, there exists a "large" set of directions Θ^* such that $\Theta^* \subset \Theta^*(x)$ for a "large" set of points $x \in R$. As a partial answer, we prove

Theorem 2. *There exists a continuous* $f : H \to W$ *such that the intersection* $\bigcap_{x \in Q} \Theta^*(x)$ *is* (a) *of the first category if* $Q \subset R$ *is residual, and* (b) *of measure zero if* $Q \subset R$ *is of full measure.*

Proof. Let Δ be as in the proof of Theorem 1, and let S be a closed subset of $H - \Delta$ that has metric density 1 at each $x \in R$. Let f be a continuous function on H with $f(\Delta) = \{0\}$ and $f(S) = \{1\}$. Then the proof of (a) is completely analogous to the proof of Theorem 1, and the proof of (b) follows the same line with property (IV) used in place of property (III).

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