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A DIRECTIONAL CLUSTER SET EXAMPLE

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§0. Introduction

Let f be a mapping from the open upper half plane H into the Riemann sphere W. For each point x on the real line R, let $C(f, x)$ and $C(f, x, \theta)$ denote respectively the total cluster set of f at x and the cluster set of f at x in the direction θ $(0 < \theta < \pi)$; then let $\Theta(x)$ denote the set of directions θ for which $C(f, x, \theta) = C(f, x)$. E. F. Collingwood [3, Theorem 2 combined with Theorem 3] established the following result.

Theorem C. *Let $f: H \to W$ be continuous. Then the set $\Theta(x)$ is residual at each point x of a residual subset S of R.*

A. M. Bruckner and Casper Goffman [2, p. 510] raised the question as to whether or not there exists a residual set of directions Θ such that $\Theta \subset \Theta(x)$ for each $x \in S$. Here we prove

Theorem 1. *There exists a continuous $f: H \to W$ such that $\bigcap_{x \in Q} \Theta(x)$ is a first category set of directions for each residual subset Q of R.*

To construct this function (§3), we use certain sets of J.-P. Kahane [5] as building blocks [1]). Two important properties of these sets are established in §2, and the necessary technical preliminaries are presented in §1. Finally, in §4 we present an example concerning essential directional cluster sets.

§1. (1/2)-trapezoids and their four descendants

By a (1/2)-*trapezoid* we mean any closed trapezoid T having bases L and L' which lie respectively on the lines $y = 0$ and $y = 1$ and for which $|L| = 2|L'|$. (Here and throughout this paper, we use $|\tau|$ to denote the length of the line segment τ.) For each (1/2)-trapezoid T, we set

$$T(1/2) = \{ z \in T : \mathrm{Im}(z) = 1/2 \}.$$

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[1]) The authors wish to thank Professor John R. Kinney for bringing this paper of J.-P. Kahane to their attention.

For real numbers p and p', let $\tau_{pp'}$ denote the line segment joining the points $(p, 0)$ and $(p', 1)$. Then the *projection of the segment* $\tau_{pp'}$ is given by

$$\operatorname{proj}[\tau_{pp'}] = p' - p,$$

and the *projection set of the* (1/2)-*trapezoid* T is given by

$$\operatorname{proj}[T] = \{\operatorname{proj}[\tau_{pp'}]: (p, 0) \in L \quad \text{and} \quad (p', 1) \in L'\}.$$

For the remainder of this section, suppose T is a (1/2)-trapezoid with bases L and L', and let L_1 and L_2 (resp., L_1' and L_2') denote the two line segments that remain when the open middle half of L (resp., L') is removed with L_1 (resp., L_1') lying to the left of L_2 (resp., L_2'). Then the *four descendants* of T are the (1/2)-trapezoids T_1, T_2, T_3, and T_4 having respective bases L_1 and L_1', L_1 and L_2', L_2 and L_1', and L_2 and L_2'.

Now let a and a' denote the respective x-coordinates of the left endpoints of L and L', and set $l = |L|$. Then the following list of facts concerning T and its descendants can easily be established:

(A) If $\hat{a} = (a+a')/2$, then

$$T(1/2) = \{(x, 1/2): \hat{a} \leqslant x \leqslant \hat{a} + 3l/4\}$$

and, for each $k = 1, 2, 3$, and 4, we have

$$T_k(1/2) = \{(x, 1/2): \hat{a} + 3(k-1)\,l/16 \leqslant x \leqslant \hat{a} + 3kl/16\}.$$

That is, the segments $T_k(1/2)$ $(k=1, 2, 3, 4)$ partition the segment $T(1/2)$ into four equal subsegments.

(B) If $\hat{v} = a' - a$, then

$$\operatorname{proj}[T] = \{v: \hat{v} - l \leqslant v \leqslant \hat{v} + l/2\},$$

$$\operatorname{proj}[T_3] = \{v: \hat{v} - l \leqslant v \leqslant \hat{v} - 5l/8\},$$

$$\operatorname{proj}[T_4] = \{v: \hat{v} - 5l/8 \leqslant v \leqslant \hat{v} - 2l/8\},$$

$$\operatorname{proj}[T_1] = \{v: \hat{v} - 2l/8 \leqslant v \leqslant \hat{v} + l/8\},$$

and

$$\operatorname{proj}[T_2] = \{v: \hat{v} + l/8 \leqslant v \leqslant \hat{v} + 4l/8\}.$$

That is, the intervals $\operatorname{proj}[T_k]$ $(k=1, 2, 3, 4)$ partition the interval $\operatorname{proj}[T]$ into four equal subintervals.