§2. The problem of resolvents

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$\xi(x, y) = \sum_{n=1}^{\infty} \frac{x^n}{n^y}$ is not a finite superposition of infinitely differentiable functions of one variable and algebraic functions of any number of variables.

The proof of this result is based on the fact that the function $\xi(x, y)$ does not satisfy any algebraic partial differential equation, that is, an equation of the form

$$\Phi\left(\xi, \frac{\partial \xi}{\partial x}, \frac{\partial \xi}{\partial y}, \ldots, \frac{\partial^{\mu+\lambda} \xi(x, y)}{\partial x^{\mu} \partial y^{\lambda}}\right) = 0, \quad \text{where} \quad \Phi$$

is a polynomial with constant coefficients in the function ξ and its partial derivatives up to a certain order. At the same, it is comparatively simple to prove that any function of two variables which is a finite superposition of infinitely differentiable functions of one variable and algebraic functions of any number of variables necessarily satisfies some algebraic partial differential equation. In the same paper, Ostrowski conjectured that the function $\xi(x, y)$ is not a superposition of continuous functions of one variable and algebraic functions of any number of variables (see the theorem of Kolmogorov [9]).

§ 2. *The problem of resolvents*

Algebraic equations up to the 4-th degree inclusive are soluble by radicals, that is, the roots of these equations can be represented as functions of the coefficients in the form of a superposition of arithmetic operations and functions of one variable of the form $\sqrt[n]{t}$ $(n = 2, 3)$. The general equation of the 5-th degree, is insoluble by radicals, as Abel and Galois showed. But since the general equation of the 5-th degree may be reduced by algebraic substitutions to the form $x^5 + tx + 1 = 0$, containing a single parameter t, we may say that a root of the general equation of the 5-th degree is also represented as a function of the coefficients in the form of superpositions of arithmetic operations and algebraic functions of one variable. The problem of resolvents can be formulated in terms of superpositions in the following way: to find, for any number n, the smallest number k such that a root of the general equation of the n-th degree as a function of the coefficients is represented in the form of a superposition of algebraic functions of k variables. In [3] Hilbert conjectured that for $n = 6, 7, 8$ the number k is 2, 3, 4, respectively. Hilbert's result [3] for an equation of the 9-th degree was all the more unexpected: a root of the general equation of the 9-th degree is representable as a superposition of algebraic functions of four variables. Wiman [13], generalizing Hilbert's result, proved that $k \leqslant n - 5$ for any $n \geqslant 9$. As G. N. Chebotarev [14] observed, it can be proved by the same method that $k \leqslant n - 6$ for $n \geqslant 21$ and $k \leqslant n - 7$ for $n \geqslant 121$. A number of papers by N. G. Chebotarev [15] was devoted to the problem of resolvents. However, the basic Theorem turned out to be wrong (see [16]).

In correcting Chebotarev's paper Morosov found the right statements but his proofs also were not without essential gaps [17]. Nevertheless, in spite of the mistakes the papers of Chebotarev and Morosov have had a positive influence on subsequent authors.

Arnol'd [18] and Lin [17] have shown that the function $f_n = f(z_1, \dots, z_n)$ which is the solution of the algebraic equation $f^n + z_1 f^{n-1} + z_2 f^{n-2} + \dots + z_n = 0$ for $n \geqslant 3$ can not be strictly represented as a superposition of entire algebraic functions of a smaller number of variables and polynomials of any number of variables. Let us recall that a function $f = f(z_1, \dots, z_k)$ is called an entire algebraic function if it satisfies an equation $f^m + p_1 f^{m-1} + \dots + p_m = 0$, where $p_1, \dots, p_m$ are polynomials in $z_1, \dots, z_k$. The sentence "a function can not be strictly represented as a superposition" means in the case under consideration that every superposition representing the function must have unnecessary branches, i.e. the number of branches of any superposition must be at least $n + 1$. Using that theorem for $n = \{ 3, 4 \}$ we see that in spite of the fact that the equations of degree 3 and 4 are soluble by radicals they do not have strict representations. This explains in a sense why unnecessary roots appear when one uses Cardano's formulas.

Hovanski (see [19] and [20]) has shown that the solution of the equation $f^5 + xf^2 + yf + 1 = 0$ can not be represented by a superposition of entire algebraic functions of a single variable and polynomials in several variables. We recall that the Tschirnhaus transformation reduces the general equation of the 5-th degree to an equation with a single parameter, that is, the function of Hovanski is represented by a superposition of algebraic functions of a single variable and arithmetic operations. This counter example demonstrates that the restriction not to use the operation of division, is really strong.

We conclude the discussion of the problem of resolvents with a formulation of a well-known problem: is it possible to represent any algebraic function by means of a superposition of functions of a single variable and rational functions of any number of variables.