§4. Superpositions of continuons functions

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **23 (1977)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **12.07.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.
Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.
Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Ein Dienst der *ETH-Bibliothek*
ETH Zürich, Rämistrasse 101, 8092 Zürich, Schweiz, www.library.ethz.ch

http://www.e-periodica.ch

the same time, from the inequalities mentioned above it follows that an increase in k leads to an insignificant improvement in the accuracy of the approximation. Hence, in a certain sense, a more economical approximation of functions is by means of expressions of the given form with $k = 1$, that is, by fractions of the form

$$\frac{\sum_{i=0}^{p} a_i f_i(x)}{\sum_{j=0}^{p} b_j g_j(x)}$$

The same inequalities with $k = 1$ show that there are no methods of approximating functions by fractions of the given form essentially better than the standard methods of approximating functions by algebraic (or trigonometric) polynomials.

§ 4. *Superpositions of continuous functions*

Kolmogorov's theorem on the possibility of representing continuous functions of n variables as superpositions of continuous functions of three variables was highly unexpected (see [7]).

In this paper Kolmogorov proves that on the n-dimensional cube $\mathscr{I}^n$ we can construct continuous functions $\varphi_i(x)$ $(i = 1, 2, \ldots, n+1)$ such that any continuous function $f(x)$, defined on the cube $\mathscr{I}^n$, can be represented in the form

$$f(x) = \sum_{i=1}^{n+1} f_i(d_i(x)),$$

where $d_i(x)$ is a continuous mapping of $\mathscr{I}^n$ onto the one-dimensional tree [1]) D if the components of the level sets of the functions $\varphi_i(x)$, and $f_i(d_i)$ is a continuous function on the tree D_i. Since the trees $\{D_i\}$ can be embedded homeomorphically in the plane (see [30]), the functions $\{f_i(d_i(x))\}$ can be thought of as superpositions

$$\{f_i(u_i(x_1, x_2, \ldots, x_n),\ v_i(x_1, x_2, \ldots, x_n))\}$$

[1]) Kronrod [29] has shown that the components of all possible level sets of any continuous function defined on $\mathscr{I}^n$ in a certain natural topology, form a tree, that is, a one-dimensional locally connected continuum, not containing homeomorphic images of circles. Kronrod calls this the "one-dimensional tree of the function".

where $\{f_i(u_i, v_i)\}$ are continuous functions of two variables, and $\{u_i(x)\}$ and $\{v_i(x)\}$ are fixed continuous functions of n variables. Kolmogorov derived from this the result that for $n \geqslant 4$ any continuous function of n variables can be represented by the following superposition of continuous functions of not more than $n-1$ variables:

$$\sum_{i=1}^{n} f_i\big(u_i(x_1, x_2, \ldots, x_{n-1}),\ v_i(x_1, x_2, \ldots, x_{n-1}),\ x_n\big).$$

Arnol'd [8], [22] showed that, firstly, in Kolmogorov's construction [7] we can manage with functions $\{\varphi_i(x)\}$ whose one-dimensional trees $\{D_i\}$ have index at each branch point equal to 3, and, secondly, for any compact set F of functions defined on such a tree D, the given tree can be so placed in three-dimensional u, v, w-space that any continuous function $f(d) = f(u, v, w) \in F$ can be represented as the sum of functions of the coordinates, $f(u, v, w) = \varphi(u) + \psi(v) + \kappa(w)$. Hence it follows that any continuous function $f(x, y, z)$ of three variables can be represented as a superposition of the form $f(x, y, z) = \sum_{i=1}^{9} f_i(\varphi_i(x, y), z)$, where all the functions are continuous, and the functions $\{\varphi_i(x, y)\}$ can be regarded as fixed, when $f(x, y, z)$ is taken from a compact set. Thus, Arnol'd had the last word in refuting Hilbert's conjecture. At the same time Kolmogorov [9] obtained, in a certain sense, the definitive result in this direction.

Each continuous function of n variables, given on the unit cube in n-dimensional space, is representable as a superposition of the form

$$f(x_1, x_2, \ldots, x_n) = \sum_{q=1}^{2n+1} g_q\Big(\sum_{p=1}^{n} \varphi_{p,q}(x_p)\Big), \tag{I}$$

where all the functions are continuous, and moreover the functions $\{\varphi_{p,q}(x_p)\}$ are standard and monotonic.

In particular, each continuous function of two variables is representable in the form

$$f(x, y) = \sum_{i=1}^{5} f_i(\alpha_i(x) + \beta_i(y)). \tag{II}$$

Kolmogorov's theorem can be supplemented by the following result of Bari, which was obtained in connection with problems of Fourier series: any continuous function of one variable $f(t)$ can be represented in the form $f(t) = f_1(\varphi_1(t)) + f_2(\varphi_2(t)) + f_3(\varphi_3(t))$, where all the functions $\{f_i\}$ and $\{\varphi_i\}$ are absolutely continuous [32].

From the theorems of Kolmogorov and Bari it follows that each continuous function of n variables can be represented as a superposition of absolutely continuous monotonic functions of one variable and the operation of addition.

A detailed account of Kolmogorov's theorem is to be found in the surveys [9], [33]-[36]. The proof presented by Kahane is of special interest [36]. He does not attempt to construct the functions $\{ \varphi_{p,q} \}$ (as the proof of Kolmogorov does) but instead he shows by means of Baire's theorem, that most selections of increasing functions $\{ \varphi_{pq} \}$ will do. This approach also lead to other interesting results.

Fridman [37] showed that the inner functions $\{ \varphi_{pq} \}$ can be chosen from the class Lip 1. Kahane noticed that this follows directly from Kolmogorov's theorem. For any finite collection of continuous and monotone functions $\{f_k(x)\}$ on the segment [0, 1] there exists a homeomorphism $x = \varphi(s)$ of the segment [0, 1] onto itself such that the functions $\{ g_k(s) = f_k(\varphi(s)) \}$ belong to the class Lip 1. The homeomorphism is taken as

$$s = \varphi^{-1}(x) = \varepsilon\big(x + \sum_k |f_k(x) - f_k(0)|\big).$$

The constant ε is chosen to satisfy the condition $\varphi^{-1}(1) = 1$. By means of such homeomorphisms all inner functions in Kolmogorov's formula can be turned into functions satisfying the condition Lip 1.

There are some other improvements of Kolmogorov's theorem: Doss [38], Bassalygo [39], Lorentz [34], Sprecher [40] (see chapter 3, § 1). There are also many results concerning special types of superpositions (see [21], [33], [41]-[44]).

§ 5. *Linear superpositions*

We return again to superpositions of smooth functions.

One of the most interesting current problems on the subject of superpositions is the following: does there exist an analytic function of two variables that cannot be represented as a finite superposition of continuously differentiable (smooth) functions of one variable and the operation of addition?

Linear superpositions arise as a result of the following argument. Suppose that a function of two variables $f(x, y)$ is an s-fold superposition of certain smooth functions of one variable $\{f_i(t)\}$ and the operation of addition. We vary this superposition, that is, we consider a superposition