§3. Theorem on superpositions of smooth functions

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§ 3. *Theorem on superpositions of smooth functions*

We will denote by $C_s(\mathscr{I}^n)$ the space of n times differentiable functions of n variables defined on the cube $\mathscr{I}^n$ with the norm

$$\|f\| = \sum_{p=1}^{s} \sum_{k_1+k_2+\dots+k_n=p} \max_{x \in \mathscr{I}^n} \left| \frac{\partial^{k_1+\dots+k_n} f(x)}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} \right|$$

THEOREM 2.3.1. *Let the numbers* $s \geqslant 1$, $s' \geqslant 1$ *and natural* n *and* n' *be such that* $\frac{n}{s} > \frac{n'}{s'}$. *Then the set of functions from* $C_s(\mathscr{I}^n)$ *not representable on* $\mathscr{I}^n$ *by superpositions of* S' *times differentiable functions of* n' *variables is a set of second category.*

The space $C_s(\mathscr{I}^n)$ is complete and consequently the set mentioned in the theorem is not empty. The theorem is true for any $s \geqslant 1$, $s' \geqslant 1$ but we will assume for simplicity that s and s' are integers.

LEMMA 2.3.1. *Let* f *and* f' *be* q*-fold superpositions composed of the functions* $\{\varphi^p_{\alpha_1,\dots,\alpha_p}\}$ *and* $\{\tilde{\varphi}^p_{\alpha_1,\dots,\alpha_p},\}$ *where all functions composing the superpositions satisfy the condition* Lip 1 *with the constant* L *and for any collection* $p, \alpha_1, \dots, \alpha_p$

$$\max \left| \varphi^p_{\alpha_1,\dots,\alpha_p} - \tilde{\varphi}^p_{\alpha_1,\dots\alpha_p} \right| \leqslant \varepsilon$$

Then

$$\max_{x \in \mathscr{I}^n} \left| f(x) - \tilde{f}(x) \right| \leqslant (L+1)^q \ \varepsilon$$

The lemma can easily be proved by induction in q.

LEMMA 2.3.2. *Let* Ω *be an open subset of* $C_s(\mathscr{I}^n)$ *and* $\Omega^* \subset C(\mathscr{I}^n)$. *If every* $f \in \Omega$ *allows uniform approximations on* $\mathscr{I}^n$ *with any accuracy by functions from* Ω^*, *i.e. the closure of* Ω^* *contains* Ω, *then* $H_\varepsilon(\Omega^*) \geqslant C\left(\frac{1}{\varepsilon}\right)^{n/s}$, *where* $C > 0$ *is independent of* ε.

The lemma is easily reduced to lemma 2.2.1 and lemma 2.2.2.

We denote by Ω_k the set of all functions of $C(\mathscr{I}^n)$ which are k-fold superpositions composed of s' times differentiable functions of n' variables with partial derivatives bounded by the same constant k.

LEMMA 2.3.3. *If* $\frac{n}{s} > \frac{n'}{s'}$ *then for any natural* k *the set* $\Omega_k \cap C_s(\mathscr{I}^n)$ *is nowhere dense in* $C_s(\mathscr{I}^n)$.

By lemma 2.3.1 and the theorem 2.2.1 for any natural k $H_\varepsilon(\Omega_k) \leqslant C\left(\frac{1}{\varepsilon}\right)^{n'/s'}$, where C does not depend on ε. Hence, it follows from the inequality $\frac{n}{s} > \frac{n'}{s'}$ and lemma 2.3.2 that the set $\Omega_k \cap C_s(\mathscr{I}^n)$ is nowhere dense in $C_s(\mathscr{I}^n)$.

Now to prove the theorem we have to notice only that the set of functions from $C_s(\mathscr{I}^n)$ representable by superpositions coincides with $\bigcup_{k=1}^{\infty} (\Omega_k \cap C_s(\mathscr{I}^n))$. By lemma 2.3.3 the sets $\{\Omega_k \cap C_s(\mathscr{I}^n)\}$ are nowhere dense and consequently the set of not representable functions is a set of second category.

CHAPTER 3. — SUPERPOSITIONS OF CONTINUOUS FUNCTIONS

In this chapter we present the proof of the theorem of Kolmogorov given by Kahane [36]. This proof which is based on Baire's theory contains a minimum of concrete constructions and shows that there exists a wide choice of inner functions for Kolmogorov's formula.

§ 1. *Certain improvements of Kolmogorov's theorem*

By the theorem of Kolmogorov any function defined and continuous on the cube $\mathscr{I}^n$ can be represented as

$$f(x_1, \ldots, x_n) = \sum_{q=1}^{2n+1} g_q\left(\sum_{p=1}^{n} \varphi_{p,q}(x_p)\right),$$

where $\{\varphi_{p,q}\}$ are specially chosen continuous and monotonic functions which do not depend on f, and where $\{g_q\}$ are continuous functions.

Lorentz [12] has noticed that in the theorem of Kolmogorov the functions $\{g_q\}$ can be chosen independently of q. In fact, by adding constants to the functions $t_q = \sum_{p=1}^{n} \varphi_{p,q}(x_p)$ $(q = 1, \ldots, 2n+1)$ one can make the ranges