

## §2. The theorem of Kahane

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2. The function  $t_q = \sum_{p=1}^n \lambda_p \varphi_q(x_p)$  maps the cube  $\mathcal{I}^n$  into the circle  $|t| = 1$ .

3. The transformation  $\Psi$  given by the equalities  $t_q = \sum_{p=1}^n \lambda_p \varphi_q(x_p)$  ( $q=1, \dots, 2n+1$ ) is one-to-one on  $\mathcal{I}^n$ .

4. For any function  $f$  continuous on  $\mathcal{I}^n$  there exists a function  $g(z)$  continuous on the disk  $|z| \leq 1$ , holomorphic inside that disk, and such that  $f = \sum_{q=1}^{2n+1} g\left(\sum_{p=1}^n \lambda_p \varphi_q(x_p)\right)$ .

The transformation  $\Psi$  gives an embedding of the cube  $\mathcal{I}^n$  into the torus  $|t| = 1$  ( $q=1, \dots, 2n+1$ ) such that any function continuous on the cube  $\tilde{\mathcal{I}}^n = \Psi(\mathcal{I}^n)$  is represented in the form  $f(t_1, \dots, t_{2n+1}) = \sum_{q=1}^{2n+1} g(t_q)$ , where  $g$  is a function holomorphic in the unit disk. This means in particular that any function continuous on  $\tilde{\mathcal{I}}^n$  has an analytic extension to the polydisk  $|t_q| \leq 1$  ( $q=1, \dots, 2n+1$ ).

## § 2. The theorem of Kahane

Let  $M$  be a complete metric space. We recall that a set is called a set of second category if it is the intersection of a countable family of open sets which are everywhere dense in  $M$ . By the theorem of Baire in a complete metric space no set of second category is empty. The massivity of such sets is characterized by the fact that the intersection of a countable family of sets of second category is again a set of second category and consequently is not empty.

We will say that a statement is true for quasi every element of  $M$  if it is true for a set of elements of second category.

Let us consider an example. Let  $\Phi$  be the space with uniform norm consisting of all functions continuous and non-decreasing on the segment  $\mathcal{I}^1$  ( $0 \leq t \leq 1$ ). It can be shown easily that quasi every element of  $\Phi$  is a strictly increasing function.

In fact, any strictly increasing function belongs to any set defined as  $\varphi(r') < \varphi(r'')$ , where  $r' < r''$  are fixed rational numbers. Any set defined by an inequality of that type is open and everywhere dense in  $\Phi$ , and the set of all such sets is countable.

Let  $\mathcal{I}^n$  be the cube  $\{0 \leq x_i \leq 1, i = 1, \dots, n\}$ ;  $C(\mathcal{I}^n)$ -the space of all functions continuous on  $\mathcal{I}^n$  with the uniform norm;  $\Phi$ -the space of functions continuous and non-decreasing on the segment  $\mathcal{I}^1$  (with the uniform norm);  $\Phi^k = \Phi \times \dots \times \Phi$  the  $k$ -th power of the space  $\Phi$ .

THEOREM 3.2.1. *Let  $\lambda_p$  ( $p=1, \dots, n$ ) be a collection of rationally independent constants. Then for quasi every collection  $\{\varphi_1, \dots, \varphi_{2n+1}\} \in \Phi^{2n+1}$  it is true that any function  $f \in C(\mathcal{I}^n)$  can be represented on  $\mathcal{I}^n$  in the form*

$$f(x) = \sum_{q=1}^{2n+1} g\left(\sum_{p=1}^n \lambda_p \varphi_q(x_p)\right),$$

where  $g$  is a continuous function.

### § 3. The main lemma

We fix a function  $f \in C(\mathcal{I}^n)$ , positive numbers  $\lambda_p$  ( $p=1, \dots, n$ ) and a positive  $\varepsilon$ . We will denote by  $\Omega_f$  the set of all collections  $\{\varphi_1, \dots, \varphi_{2n+1}\} \in \Phi^{2n+1}$  for each of which there exists a continuous function  $h$  such that  $\|h\| \leq \|f\|$  and  $\|f(x) - \sum_{q=1}^{2n+1} h\left(\sum_{p=1}^n \lambda_p \varphi_q(x_p)\right)\| < (1-\varepsilon)\|f\|$ . The latter inequality is strict and consequently the set  $\Omega_f$  is open.

The idea of the construction is contained in the following statement.

LEMMA 3.3.1. *If  $\|f\| \neq 0$ , the numbers  $\{\lambda_p\}$  are rationally independent, and  $0 < \varepsilon < \frac{1}{2n+2}$ , then the corresponding set  $\Omega_f$  is everywhere dense in  $\Phi^{2n+1}$ .*

*Proof.* Let us fix an open set  $\Omega \subset \Phi^{2n+1}$  and prove that  $\Omega \cap \Omega_f$  is not empty. This will imply that  $\Omega_f$  is everywhere dense in  $\Phi^{2n+1}$ .

We choose a number  $\delta > 0$  and denote by  $\mathcal{I}_q(j)$  the segment defined by the inequality

$$q \cdot \delta + (2n+1)j \cdot \delta \leq t \leq q \cdot \delta + (2n+1)j\delta + 2n\delta$$

( $q=1, \dots, 2n+1, j$  is an integer).

The value  $\delta$  will be determined below. Now we notice, firstly, that for any  $q$  the segments  $\mathcal{I}_q(j)$  ( $j=0, \pm 1, \pm 2$ ) are pairwise disjoint and every two consecutive segments are separated by an interval of length  $\delta$  and, secondly, that, every point of the real axis belongs to at least  $2n$  of the sets  $\sum_j \mathcal{I}_q(j)$ , ( $q=1, \dots, 2n+1$ ).